Mathematical Physics Studies

Alexander Cardona
Andrés F. Reyes Lega *Editors*

Geometry, Topology and Operator Algebras

Global Analysis, Invariants and Their Significance in Theoretical Physics

Springer

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Alexander Cardona · Andrés F. Reyes Lega
Editors

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Global Analysis, Invariants and Their Significance in Theoretical Physics

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Preface

Geometry, so far in the twenty-first century, has been increasingly enriched with tools from other areas of mathematics, such as algebra and analysis. This line of development has of course permeated theoretical physics, at the same time a source of examples and motivations for these and other developments in modern mathematics, making tools specific to operator algebras and functional analysis to appear with increasing relevance in the description of physical systems in which geometry and topology play an important role. The present volume has its origin in the lecture notes of short courses—taught by specialists in global analysis, operator algebras, topological invariants and their applications to theoretical physics—given during two summer schools (held in 2017 and 2019 in Villa de Leyva), on topics lying on the borderline between geometry, topology, algebra and their use in theoretical physics.

The different chapters of this volume present a coherent and updated overview of recent developments in this broad field, touching on such important topics as the geometry and topology of foliations, the geometry and topology of affine varieties, the hyperbolic and homogeneous geometry of non-commutative C^*-algebras, the pseudo-differential calculus associated with geometric structures and spectral problems with boundary values, and the use of these many tools of global analysis in the description of topological matter. All chapters are intended for graduate students and researchers alike. They aim to bring the reader through the contemporary methods of global analysis, operator algebras, and their use in the computation of invariants of geometric and topological nature whose significance in theoretical physics has been clearly established by now.

The goal of the first chapter "Foliations and Operator Algebras", Georges Skandalis, is to introduce foliations, some algebraic structures associated to them (groupoids and operator algebras) and the longitudinal index theorem for these geometric structures. Foliations are manifolds equipped with an equivalence relation whose equivalence classes, called the leaves, are connected submanifolds. After a quick introduction to the differential geometry of manifolds and vector bundles, the reader will find in this chapter a survey of C^*-algebras, which can be regarded as the non-commutative analogues of the (commutative) algebras of continuous functions on locally compact topological spaces. C^*-algebras and their geometry are

also the subject of the contributios to this volume by Esteban Andruchow, Gustavo Corach and Lázaro Recht. The two chapters by these authors include the study of multiple non-commutative geometric structures over the set of positive elements in these non-commutative operator algebras (bringing the well-known homogeneous and hyperbolic structure of positive elements in the complex numbers to the non-commutative world). The introductory sections to C^*-algebras and the K-theory of operator algebras by Skandalis will also certainly be of great benefit to readers of other chapters of this volume in which these objects are studied.

A way in which we can associate a noncommutative C^*-algebra to a foliation is through groupoids, structures which generalize both spaces and groups and are particularly suited to study the geometry of singular spaces. Definitions, examples and the main properties of Lie groupoids are considered in this volume, in particular Connes' tangent groupoid and its connection with the analytic index map used to build topological invariants of foliations. In the first chapter "Foliations and Operator Algebras", the way the definition of invariants associated to foliations is given, is by the use of algebras of pseudo-differential operators associated to the holonomy groupoid of the foliation, which in turn gives rise to the longitudinal index theorem for foliations (using operators that are elliptic along the leaves of the foliation). The Fredholm index of an elliptic pseudo-differential operator is the difference between the dimension of its kernel and the codimension of its image, and the K-theory of the C^*-algebra of the foliation plays the role of the group of (formal differences of) dimensions of certain finitely generated projective modules, from which the longitudinal index theorem for foliations can be stated. This important result is the main theme and goal of the first chapter "Foliations and Operator Algebras", rich in tools and examples which motivate most of the remaining content of the volume.

In their "Lectures on the Euler Characteristic of Affine Manifolds" (the second chapter), Camilo Arias Abad and Sebastián Vélez Vásquez aim to provide an introduction to a conjecture by Shiing-Shen Chern, which states that the Euler characteristic of a closed affine manifold vanishes, showing the motivation for the conjecture, many results of the last seventy years around the conjecture, as well as some recent developments. Recall that, given a closed Riemannian manifold (M, g) with Levi-Civita connection ∇, the Chern-Gauss-Bonnet formula gives a way to compute the Euler characteristic of M from the curvature of ∇, namely:

$$\chi(M) = \frac{1}{(2\pi)^n} \int_M \mathrm{Pf}(K).$$

This result, which actually is a particular case of the Atiyah-Singer index formula for the Dirac operator, cannot be used to give an answer to Chern's conjecture on the Euler characteristic of a closed affine manifold because, in the case of manifolds which admit flat torsion-free connections on the tangent bundle, a compatible metric cannot be guaranteed. The contribution by Arias and Vélez contains the basic material necessary to give a self-contained introduction to Chern's conjecture to students at the graduate level and researchers, including appendices with preliminary material on the

geometry of connections, curvature and the Chern-Gauss-Bonnet theorem, spectral sequences and sheaf theory. The reader will find in this chapter an exposition of results concerning the Euler characteristic of affine manifolds which include Milnor's inequalities and Benzécri's proof of the conjecture for closed affine surfaces, the proof by Kostant and Sullivan in the case where the manifold is complete and Smillie's work showing that the condition that the connection is torsion-free matters. Finally, much more recent is the proof that the conjecture is true for affine manifolds that admit a parallel volume form made by Klingler, using methods of sheaf theory and spectral sequences.

The contribution by Esteban Andruchow, Gustavo Corach and Lázaro Recht to this volume is twofold. First, in "Homogeneous Spaces of Operators" (the third chapter), the authors study the homogeneous geometry of the space of positive elements in a C^*-algebra $\mathcal{A}$. In a unital C^*-algebra $\mathcal{A} = \mathcal{A}_h \oplus \mathcal{A}_{ah}$, decomposed in terms of hermitian and anti-hermitian elements, the starting point of the construction is the set of positive and invertible elements in $\mathcal{A}$, denoted by G^+, which sits inside $G_{\mathcal{A}}$ (the space of invertible elements), which has a rich geometry coming from the reductive homogeneous structure given by the natural isotropies defined by $\mathcal{U}_{\mathcal{A}}$, the group of unitary elements in $\mathcal{A}$. The authors begin by describing the differential geometry of G^+, for which the main input is the natural covariant derivative associated to the horizontal distribution (given by the background decomposition $\mathcal{A} = \mathcal{A}_h \oplus \mathcal{A}_{ah}$ in the C^*-algebra). The notions of parallel transport and geodesic give a natural metric structure in the noncommutative space G^+. Using the polar decomposition allowing to regard sets of idempotents as subsets of G^+, many different presentations of homogeneous spaces (of the unitary group) are considered, which will play a central role in the construction of geometric structures (complex, hyperbolic, symplectic, among many others) on the aforementioned homogeneous spaces.

In the fourth chapter entitled "The Poincaré Half-Space of a C^*-Algebra", Andruchow, Corach and Recht continue their study of the hyperbolic space associated to the positive elements in a C^*-algebra, and its alternative presentations. Regarded as the noncommutative analogue of the Poincaré disc, the projective nature of this space arises and geometric properties (as the limit points of geodesics) can be conveniently studied, like the operator valued cross-ratio of a four-tuple induced by a geodesic, which coincides with what happens in the classic scalar Poincaré disk. The chapter goes on with the study of the analogue of the convexity result for the image of the moment map, a translation of the classical result of Atiyah, Guillemin and Sternberg to this noncommutative setting. Indeed, the authors show that there is a natural operator-valued Kähler structure on the Poincaré half-space, inducing a homogeneous symplectic structure, which is the imaginary part of the Hilbertian form. The natural action of unitaries gives rise to a moment map which is exhibited explicitly. The chapter ends with the study of the projective structures induced by these homogeneous fibrations.

In the fifth chapter "An Introduction to the Non-harmonic Analysis of Pseudo-Differential Operators", Alexander Cardona and Michael Ruzhansky address recent progress in the study of the *non-harmonic analysis of pseudo-differential operators*, meaning by that the symbolic calculus and basic algebraic and analytic

properties of pseudo-differential operators, defined by and adapted to spectral information of a non-necessarily self-adjoint elliptic differential operator. This algebra of global pseudo-differential operators is associated to the boundary value problem on a domain Ω, associated to the spectral data furnished by a differential operator $\mathfrak{L}$ with fixed boundary conditions, giving rise to biorthogonal systems of eigenfunctions in $L^2(\Omega)$. In the first half of the chapter, the non-harmonic Fourier series are introduced, as well as the Riesz basis and other important tools necessary to define pseudo-differential operators on spaces with no prescribed symmetries. In the case of smooth manifolds with symmetries, like Lie groups or homogeneous spaces, a global pseudo-differential calculus can be carried on using the tools of representation theory, from symbols with values in matrices, which is briefly reviewed in the text.

The goal of the non-harmonic analysis when defining pseudo-differential operators is to show that those methods can be equally translated to situations in which no symmetries are involved, using as a reference the spectral information of a non-necessarily self-adjoint elliptic differential operator acting on a domain with specific boundary conditions. In the second part of the fifth chapter "An Introduction to the Non-harmonic Analysis of Pseudo-Differential Operators", the notions of Fourier transform and Sobolev spaces are adapted to the reference operator, and results as the Plancherel formula are shown to hold under natural spectral assumptions on $\mathfrak{L}$. The difference operators and other tools used to obtain a global symbolic calculus for pseudo-differential operators associated to boundary value problems are introduced. As expected, when the elliptic differential operator $\mathfrak{L}$ used in the construction is the Laplacian, the associated pseudo-differential calculus coincides with the regular one. It is worth mentioning that the techniques here generalized are the very same used in, for example, the first chapter on "Foliations and Operator Algebras" to define topological invariants associated to geometric structures. Also, the role of boundary conditions associated to differential or pseudo-differential operators also meets topology in many applications in physics, for example in the ones considered by Asorey and Martínez in the last chapter of this volume.

Manuel Asorey and Yisely Martínez, in their lectures on "Mathematical Foundations of Topological Matter" (the sixth chapter), introduce the description of certain quantum states of matter, characterized by topological invariants, which are important for potential applications. This last chapter delves into key concepts such as quantum phase transitions, the topology of metallic bands, edge states, and the bulk-edge correspondence. The authors guide the reader through an understanding of how these topological features manifest in systems like graphene, with an emphasis on the Integer Quantum Hall Effect and its variations. By providing a rigorous mathematical framework, this chapter bridges the gap between abstract topological ideas and their concrete physical applications, offering a rich source of tools for those interested in the intersection of geometry, topology, and condensed matter physics.

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Bogotá, Colombia

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Contents

Foliations and Operator Algebras 1
Georges Skandalis

Lectures on the Euler Characteristic of Affine Manifolds 59
Camilo Arias Abad and Sebastián Vélez Vásquez

Homogeneous Spaces of Operators 121
Esteban Andruchow, Gustavo Corach, and Lázaro Recht

The Poincaré Half-Space of a C^*-Algebra 169
Esteban Andruchow, Gustavo Corach, and Lázaro Recht

An Introduction to the Non-harmonic Analysis of Pseudo-Differential Operators 217
Alexander Cardona and Michael Ruzhansky

Mathematical Foundations of Topological Matter 249
Manuel Asorey and Yisely Martínez

Index 279

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Foliations and Operator Algebras

Georges Skandalis

Abstract This chapter introduces the concepts of foliations and their associated operator algebras, with the aim of discussing the longitudinal index theorem for foliations. Fundamental concepts necessary to achieve this goal are introduced in the first sections. This includes notions of differential geometry, operator algebras and K-theory. Then, the notions of Lie groupoids and their C^*-algebras are introduced. This is followed by a discussion of pseudodifferential operators on manifolds and their extension to the case of Lie groupoids. The remaining sections are devoted to foliations, their holonomy groupoid and an outline of the proof of the longitudinal index theorem for foliations.

1 Introduction

The aim of this chapter is to introduce foliations, their operator algebras and longitudinal index theory. We will illustrate this by stating the longitudinal index theorem for foliations of [9].

A *foliation*, or *foliated manifold*, is a manifold M equipped with an equivalence relation whose equivalence classes, called the *leaves*, are connected submanifolds. We assume that locally the picture is a product $\mathbb{R}^p \times \mathbb{R}^q$—with local connected components of the leaves (called *plaques*) $\mathbb{R}^p \times \{u\}$ with $u \in \mathbb{R}^q$—here $\mathbb{R}^q$ is the *local transversal.*

We will give a precise definition of foliations and typical examples, related with actions of groups, and Reeb type foliations obtained by gluing foliations on manifolds with boundaries. We will outline the Frobenius theorem which describes foliations by first order data: the set of vectors tangent to the leaves. Then, we will introduce the notion of *holonomy* of a foliation, and construct the *holonomy groupoid.*

A foliation appears naturally when one studies phenomena which propagate only in some directions of the space: along the leaves. Mathematically, this propagation

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can be written using partial differential equations of the form $\frac{\partial f}{\partial t} = Df$ where $f : (t, x) \mapsto f_t(x)$ is a function on $\mathbb{R} \times M$, and we study the time evolution $t \mapsto f_t$ of this function. The operator D, is a *differential operator* on M, i.e. an operator using partial derivatives of f in local coordinates of M. The longitudinal ones will only use the partial derivatives along the coordinates of the leaves.

The celebrated *Atiyah-Singer index theorem* deals with such differential operators which are *elliptic* i.e. differentiate 'equally in all directions'. If D is an elliptic operator on a compact manifold, then its kernel has finite dimension and its image has finite codimension. The *index* of D is the difference

$$\operatorname{ind} D = \dim(\ker D) - \operatorname{codim}(\operatorname{im} D) = \dim(\ker D) - \dim(\ker D^*)$$

between the dimension of its kernel and the codimension of its image, i.e. the dimension of the kernel of the adjoint of D. The index theorem computes this index in terms of 'topological data'.

The longitudinal index theorem for foliations deals with operators that differentiate only in the directions of the leaves and are elliptic along the leaves. The kernel of such an operator is no longer finite dimensional. But there is still an index which lives in an abelian group: the K-theory of the associated C^*-algebra.

Somewhat loosely speaking, the kernel and cokernel of a longitudinal elliptic operator are not finite dimensional vector spaces, but they are 'nice modules' over the C^*-algebra, i.e. finitely generated projective modules. The K-theory of the C^*-algebra plays then the role of the group of (formal differences of) dimensions of these finitely generated projective modules.

The C^*-algebra associated with the foliation is constructed thanks to its *holonomy groupoid*. This is a particular case of a *Lie groupoid*. Lie groupoids are natural objects generalizing Lie groups, and on which there is a natural notion of convolution. The notion of convolution appears in many settings, and this rather familiar operation is very much related with groupoids. Using the convolution, we may associate a C^*-algebra to any Lie groupoid, and in particular to the holonomy groupoid of a foliation.

So, in order to reach the statement of the longitudinal index theorem for foliations, and say a few words on its proof, we will need to introduce many notions.

In the first three sections, we will visit various areas of mathematics:

- differential geometry with the notion of *manifolds*;
- functional analysis for the definition and first properties of *C^*-algebras*;
- algebraic topology and in particular *K-theory*—we will only introduce the K-theory of (complex) Banach algebras and C^*-algebras.

The next two sections will be devoted to the notion of *Lie groupoids* and their C^*-algebras.

Finally, we will use some theory of (linear) partial differential equations and global analysis: we will say a few words on pseudodifferential operators on a manifold, and then extend this notion to the case of Lie groupoids.

After this general material, we will come to a shorter second part devoted to the main theme of our text: foliations, their holonomy groupoid and index theorems.

On the References

The main notions on foliations and holonomy groupoids can be found in [6, 20].
Basics on Banach algebras can be found in [24].
The relation of groupoids with index theory is very well explained in [12].
The main notions on K-theory of Banach algebras can be found in [2, 16].
For those who want to learn more about C^*-algebras, operator K-theory and in particular Kasparov's bivariant K-theory and variations [5, 14, 26, 28].
A general overview of the subject—and much more—in Connes' book: [8].

On the Notation

We will use the famous *Alfonso's notation*:

- $*$ we will put a $(*)$ in front of a statement which is willingly not too precise;
- $\star$ we will put a $[\star\star]$ in front of a statement that we admit.

2 Some General Notions

We start by some classical notions on differential manifold, operator algebras and K-theory. We will only give very few information on each of these subjects—more or less only as much information as we need in the main part.

2.1 Manifolds

2.1.1 Charts and Smooth Manifolds

A smooth manifold of dimension n is a space which locally looks like $\mathbb{R}^n$ closely enough so that we can define smooth functions on it. Indeed, to see whether a function is smooth, you just need to know that it is smooth near every point (a locality property) and, around every point, our manifold can be identified with $\mathbb{R}^n$.

We will give a formal definition based on *charts.*

It will be convenient to use the following notation on the composition of partially defined maps.

Notation (Partial maps) Let X, Y, Z be sets.

1. A *partial map* from X to Y is a map $f : \mathrm{dom}(f) \to Y$ where $\mathrm{dom}(f)$ is a subset of X.
2. The graph of a partial map f from X to Y is the subset $\{(x, f(x));\ x \in \mathrm{dom}(f)\}$ of $X \times Y$.
3. Let f_1, f_2 be partial maps from X to Y. We write $f_1 \subset f_2$ if $\mathrm{dom}(f_1) \subset \mathrm{dom}(f_2)$ and, for every $x \in \mathrm{dom}(f_1)$ we have $f_1(x) = f_2(x)$.
4. Given an *injective* partial map f from X to Y, we define f^{-1} from Y to $X$with domain $f(\mathrm{dom}(f))$ defined by $f^{-1}(f(x)) = x$ (for $x \in \mathrm{dom}(f)$).
5. Let f be a partial map from X to Y and g be a partial from Y to Z. We define the composition $g \circ f$ to be the partial map from X to Z with domain $\{x \in X;\ f(x) \in \mathrm{dom}(g)\}$ and $g \circ f(x) = g(f(x))$.

Definition 1 A smooth manifold of dimension $n \in \mathbb{N}$ is a set M endowed with a family (indexed by a set I) of

- subsets $(\Omega_i)_{i\in I}$ such that $\bigcup_{i\in I} \Omega_i = M$;
- bijections $\varphi_i : \Omega_i \to O_i$ where O_i is a an open subset of $\mathbb{R}^n$ called *charts.*

We assume that, for every pair $(i, j) \in I^2$, the subset $O_{i,j} = \varphi_i(\Omega_i \cap \Omega_j)$ is *open* in $\mathbb{R}^n$ and that the bijection $\varphi_i \circ \varphi_j^{-1} : O_{j,i} \to O_{i,j}$ is a *smooth diffeomorphism.*

A manifold has a topology: a set $U \subset M$ is open if for every $i \in I$ the subset $\varphi_i(U \cap \Omega_i)$ is open in $\mathbb{R}^n$. Then, given any topological space X, a map $f : M \to X$ is continuous if and only if $f \circ \varphi_i^{-1}$ is continuous.

Definition 2 Let M be a smooth manifold whose smooth structure is given by $(\Omega_i, \varphi_i)_{i\in I}$.

1. Let $k \in \mathbb{N} \cup \{\infty\}$. A map $f : M \to \mathbb{R}$ is said to be of class C^k if $f \circ \varphi_i^{-1}$ is of class C^k for every $i \in I$. We denote by $C^k(M)$ the algebra of maps of class C^k.
2. Let N be a smooth manifold. A map $f : M \to N$ is of class C^k if $g \circ f$ is of class C^k for every map $g : N \to \mathbb{R}$ of class C^k.

Let M, N be manifolds whose smooth structure is respectively given by $(\Omega_i, \varphi_i)_{i\in I}$ and $(U_j, \psi_j)_{j\in J}$. Let $f : M \to N$ be a map. For $i \in I$ and $j \in J$, let $W_{i,j} = \{x \in \Omega_i;\ f(x) \in U_j\}$, and $f_{i,j} : \varphi_i(W_{i,j}) \to \psi_j(U_j)$ the map $x \mapsto \psi_j(f(\varphi_i^{-1}(x)))$.

- The map f is continuous, of class C^k, or smooth if and only for every i, j, the set $\varphi_i(W_{i,j})$ is open (in $\mathbb{R}^m$—where m is the dimension of M) and the map $f_{i,j}$ is continuous, of class C^k, or smooth.
- Assume f is of class C^1; then f is said to be a *submersion* (*resp.* an *immersion*) if the maps $f_{i,j}$ are submersions (*resp. immersions*), i.e. their differential at each point is surjective (*resp.* injective).

Definition 3 Let M be a smooth manifold. A smooth *submanifold* of M is a locally closed subset V of M, which is endowed with a smooth structure in such a way that the inclusion $i : V \to M$ is a smooth immersion.

Equivalently, this means for that every $x \in V$, there an open neighborhood Ω of x in M and a smooth submersion $f : \Omega \to \mathbb{R}^k$ such $V \cap \Omega = f^{-1}(\{f(x)\})$.

Remark 1 With our definition, a manifold doesn't need to be Hausdorff. Recall that a topological space X is *Hausdorff* if every two points $x, y \in X$ (with $x \neq y$) can be separated by disjoint open sets U, V of X, with $x \in U$ and $y \in V$.

In fact, one easily proves:

Proposition 1 *The topological space M is Hausdorff if and only if, for every i, j, the subset $\{(x, \varphi_{i,j}(x));\ x \in O_{j,i}\}$ of $O_i \times O_j$ is closed.*

Remark 2 One can even construct the set M out of the *cocycle* $(O_{i,j}, \psi_{i,j})$. Assume indeed that we are given

- a family $(O_i)_{i\in I}$ of open subsets of $\mathbb{R}^n$;
- for every pair $(i, j) \in I^2$ an open subset $O_{i,j}$ of O_i with $O_{i,i} = O_i$.
- for every pair $(i, j) \in I^2$ a diffeomorphism $\psi_{i,j} : O_{j,i} \to O_{i,j}$;
- for every triple $(i, j, k) \in I^3$, the composition $\psi_{ij} \circ \psi_{jk}$ is *contained* in ψ_{ik}. In other words, for every $x \in \psi_{k,j}(O_{j,k} \cap O_{j,i})$ we have $x \in O_{k,i}$ and $\psi_{i,k}(x) = \psi_{i,j}\big(\psi_{j,k}(x)\big)$ (cocycle condition).

Then we may define an equivalence relation on $\coprod_{i\in I} O_i = \{(i, x) \in I \times \mathbb{R}^n;\ x \in O_i\}$ by writing $(i, x)\ R\ (j, y)$ if $y \in O_{j,i}$ and $x = \psi_{i,j}(y)$. Then M is the quotient of this equivalence relation.

Comment. If we only require the maps $\psi_{i,j}$ to be of class C^k, we obtain a manifold of class C^k. On the other hand, an extra requirement on the maps $\psi_{i,j}$ endow the manifold M with an extra structure. For instance,

- if the $\psi_{i,j}$'s are (real) analytic, then M is a (real) analytic manifold;
- if they are affine maps, we obtain an affine manifold (see the notes of Camilo Arias Abad);
- if $O_{i,j} \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ and the $\psi_{i,j}$'s are holomorphic, then M is a complex manifold.

The definition of foliations that we will see is an other example of this extra requirement on the chart changes.

2.1.2 Vector Bundles

Real vector bundles. A smooth *real vector bundle* over a manifold M is a smooth manifold E with a smooth submersion $p : E \to M$. We assume that, for every $x \in M$, the fiber $E_x = p^{-1}(x)$ is endowed with a structure of a real finite dimensional vector space. Moreover, we assume *local triviality*, i.e. that there is an open cover $(\Omega_i)_{i\in I}$ of M such that $p^{-1}(\Omega_i)$ diffeomorphic to $\Omega_i \times \mathbb{R}^n$ with the usual vector space structure on the fiber $\{x\} \times \mathbb{R}^n \simeq \mathbb{R}^n$.

In terms of chart changes, assume that the bundle E is trivial over open sets Ω_i with charts $\varphi_i : \Omega_i \to O_i \subset \mathbb{R}^m$ for M. The chart changes $\tilde{\psi}_{i,j} = \tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1}$ for E are of the form: $(x, \xi) \mapsto (\psi_{i,j}(x), T_x(\xi))$ where $x \mapsto T_x$ is a smooth map from $O_{j,i}$ to $GL_n(\mathbb{R})$.

Complex vector bundles. We define in the same way *complex vector bundles* by replacing real vector spaces by complex vector spaces. They are defined by chart changes $\widetilde{\psi}_{i,j} : O_{j,i} \times \mathbb{C}^n \to O_{i,j} \times \mathbb{C}^n$ of the form $(x, \xi) \mapsto (\psi_{i,j}(x), T_x(\xi))$ where $x \mapsto T_x$ is a smooth map from $O_{j,i}$ to $GL_n(\mathbb{C})$.

Isomorphism of vector bundles. An isomorphism of (real or complex) vector bundles $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ is a smooth diffeomorphism $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$ and, for every $x \in M$, the restriction of $f : (E_1)_x \to (E_2)_x$ is linear.

Fibered product of vector bundles. If $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ are vector bundles, their *fibered product* $E_1 \times_M E_2 = \{(\xi_1, \xi_2) \in E_1 \times E_2;\ p_1(\xi_1) = p_2(\xi_2)\}$ is a vector bundle. The fiber over $x \in M$ is $(E_1)_x \times (E_2)_x$.

Dual vector bundle. If $p : E \to M$ is a (real or complex) vector bundle, we may define the dual bundle E^*. The fiber at $x \in M$ is $(E_x)^*$. In terms of chart changes, if E is given by chart changes $\widetilde{\psi}_{i,j} : O_{j,i} \times \mathbb{C}^n \to O_{i,j} \times \mathbb{C}^n$ of the form $(x, \xi) \mapsto (\psi_{i,j}(x), T_x(\xi))$ from $O_{j,i}$ to $GL_n(\mathbb{C})$, then E^* is given by the chart changes $\widehat{\psi}_{i,j} : O_{j,i} \times \mathbb{C}^n \to O_{i,j} \times \mathbb{C}^n$ given by $(x, \xi) \mapsto (\psi_{i,j}(x), {}^tT_x^{-1}(\xi))$.

Sections of a vector bundle. Let $p : E \to M$ be a (real or complex) vector bundle. A smooth section of E is a smooth map $s : M \to E$ such that $p \circ s$ is the identity of M. The set of sections of E is denoted by $\Gamma(E)$. It is obviously a vector space. If $f \in C^\infty(M)$ and $s \in \Gamma(E)$, we define the section fs by putting $(fs)(x) = f(x)s(x) \in E_x$. Thus $\Gamma(M)$ is a *module* over $C^\infty(M)$.

The notation s_x rather than $s(x)$ for the value of s at the point x is often used.

Let us note that if $s \in \Gamma(E)$ is a smooth section of a bundle E and $\xi \in \Gamma(E^*)$ is a smooth section of its dual bundle, then the map $x \mapsto \xi_x(s_x)$ is a smooth function on M.

2.1.3 Tangent Vectors, Tangent Bundle, Vector Fields

Tangent Vectors

A *tangent vector* at a point x of manifold M can be defined in various equivalent ways. We chose here to present it as a *derivative.*

Definition 4 Let M be a manifold and $x \in M$. A tangent vector to M at x is a linear map $\xi : C^\infty(M) \to \mathbb{R}$ satisfying $\xi(fg) = \xi(f)g(x) + f(x)\xi(g)$. We denote by T_xM the set of tangent vectors to M at x.

Obviously T_xM is a vector subspace of the space of linear maps $C^\infty(M) \to \mathbb{R}$. Let $\xi \in T_xM$ and $f \in C^\infty(M)$. It is not difficult to see that

1. If $f = f_1 f_2$ with $f_1(x) = f_2(x) = 0$, then $\xi(f) = 0$.
2. If f is the constant function equal to 1, then $\xi(f) = 0$.
   Indeed, write $f^2 = f$; it follows that $\xi(f) = f(x)\xi(f) + \xi(f)f(x) = 2\xi(f)$.
3. If $f(y) = 0$ for y in a neighborhood of x, then $\xi(f) = 0$.
   Indeed there exists a (bump) function $g \in C^\infty(M)$ such that $g(x) = 1$ but $g(y) = 0$ for y in the support of f. Then $fg = 0$ and it follows that $0 = \xi(fg) = \xi(f)g(x) + f(x)\xi(g) = \xi(f)$ since $f(x) = 0$ and $g(x) = 1$.

Let then U be an open subset of M that we identify with an open subset V of $\mathbb{R}^n$. For $x \in U \simeq V$ and a tangent vector $\xi \in T_xM$, the value $\xi(f)$ only depends on the restriction of f to U. We thus find that $T_xM = T_xU \simeq T_xV = T_x\mathbb{R}^n$.

A smooth fonction f can be written as

$$f(y_1, \ldots, y_n) = f(x) + \sum_{j=1}^{n} (y_j - x_j) \frac{\partial f}{\partial x_j}(x) + g(y_1, \ldots, y_n)$$

where g is a sum of products of two functions vanishing at x. It follows that $\xi(f) = \sum_{i=1}^{n} \xi(p_i) \frac{\partial f}{\partial x_i}(x)$, where p_i is the function $(y_1, \ldots, y_n) \mapsto y_i$ (note that the function $q_i : (y_1, \ldots, y_n) \mapsto x_i$ is constant, and therefore $\xi(q_i) = 0$). Therefore, a basis of T_xV is the family $f \mapsto \frac{\partial f}{\partial x_i}(x)$ $(1 \leq i \leq n)$. In particular, T_xM is isomorphic with $\mathbb{R}^n$.

Tangent Vector to a Curve

Let $\gamma : J \to M$ be a smooth *curve* drawn in M, where J is an open interval. Let $a \in J$. The curve γ defines a tangent vector $\gamma'(a) = L$ on $\gamma(a)$ by setting $L(f) = (f \circ \gamma)'(a)$ for every $f \in C^\infty(M)$.

This construction leads to an equivalent definition of tangent vectors: they are equivalence classes of pairs (γ, a) where $\gamma : J \to M$ is a smooth curve drawn in

M and $a \in J$ for an equivalence relation saying that (γ_1, a_1) and (γ_2, a_2) define the same vector if $\gamma_1(a_1) = \gamma_2(a_2)$ and, identifying a neighborhood of $\gamma_1(a_1)$ in M with an open $\mathbb{R}^n$ using any chart around this point, $\gamma_1(a_1 + t) = \gamma_2(a_2 + t) + o(t)$.

Tangent Bundle

It follows that if M is a manifold of dimension n, then $T_x M$ is a vector space of dimension n for every x. Even better, for every open subset M_1 of M diffeomorphic to an open subset of $\mathbb{R}^n$, there is a way to choose a basis for every $T_x M$ for $x \in M_1$. It follows that TM is a (real) vector bundle over M.

Differential of a Smooth Map: $M \to N$

Given manifolds M and N a map $h : M \to N$ is *smooth* if it is smooth on every chart. More precisely, this means that, for every chart $\psi : V \subset N \to U \subset \mathbb{R}^n$ of N, the map $x \mapsto \psi(f(x))$ defined on $h^{-1}(V)$ is smooth.

Given a smooth map $h : M \to N$, a point $x \in M$ and a tangent vector $\xi \in T_x M$, we denote by $dh(\xi) \in T_{h(x)} N$ the element given by $dh(\xi)(f) = \xi(f \circ h)$.

In this way we have a map $dh : TM \to TN$ called the *differential* of h. It is linear on the fibers $T_x M \to T_{h(x)} N$.

Let us also add some definitions

- A bijection $f : M \to N$ is a *diffeomorphism* if both f and f^{-1} are smooth
- A smooth map $f : M \to N$ is an *immersion* if, for every $x \in M$, its differential $dh_x : T_x M \to T_{h(x)} N$ is injective.
- A smooth map $f : M \to N$ is a *submersion* if, for every $x \in M$, its differential $dh_x : T_x M \to T_{h(x)} N$ is surjective.

Vector Fields

A smooth section of this bundle assigns to each x a tangent vector ξ_x in a smooth way, which means that for every $f \in C^\infty(M)$ the map $L : x \mapsto \xi_x(f)$ is smooth. Those sections are called *vector fields.*

From the definition of $T_x M$, it follows immediately that a vector field is a map $L : C^\infty(M) \to C^\infty(M)$ such that, for every $f, g \in C^\infty(M)$ we have $L(fg) = L(f)g + fL(g)$. Conversely every derivation of $C^\infty(M)$ is given by a vector field.

If X, Y are vector fields, the map the map $X \circ Y - Y \circ X$ from $C^\infty(M)$ into itself is easily seen to be a derivation. We denote by $[X, Y]$ the corresponding vector field. If f is a smooth function, we have $[X, fY] = f[X, Y] + (X(f))[Y]$.

Flow of a Vector Field

Given a vector field X in a manifold M, there is a well defined *flow*, which is a smooth map Φ, defined on an open subset W of $M \times \mathbb{R}$ with values in M. It satisfies the conditions:

- For every $x \in M$, the set $\{t \in \mathbb{R};\ (x,t) \in W\}$ is an interval of $\mathbb{R}$ containing 0; we have $\Phi(x,0) = x$;
- for every $(x,t) \in W$, the curve $\gamma_x : s \mapsto \Phi(x,s)$ is tangent at t to $X(\Phi(x,t))$, i.e. $\gamma_x'(t) = X(\Phi(x,t))$.

For $x \in M$, the function γ_x is called the *integral curve* of X through x. Note that these curves determine Φ.

In coordinates (i.e. if M is an open subset of $\mathbb{R}^n$), the function γ_x is the solution on a Cauchy problem: $y'(t) = X(y(t))$ and $y(0) = x$. It follows (using the Cauchy-Lipschitz theorem on differential equations) that (a maximal) Φ is uniquely defined.

For $t \in \mathbb{R}$ (small), the map $x \mapsto \Phi(x,t)$ is a diffeomorphism (between open sets) of M, denoted by $\exp_{tX}$.

If the vector fields X, Y commute, i.e. if $[X,Y] = 0$, then $\exp_{tX}$ and $\exp_{sY}$ commute. If they only commute locally in an open set $U \subset M$, then for every compact subset of U there exists $\varepsilon > 0$ such that, if $|s|, |t| \leq \varepsilon$ small enough, we have $\exp_{tX} \exp_{sY}(x) = \exp_{tY} \exp_{sX}(x)$.

Cotangent Bundle

The dual bundle T^*M of the tangent bundle TM of a manifold is called the *cotangent bundle*. Note that, for every $f \in C^\infty(M)$, we may assign a section of T^*M denoted by df defined by the formula $df_x(\xi) = \xi(f)$ for $x \in M$ and $\xi \in T_xM$. The sections of T^*M are called 1-*forms* on M; the 1-form df is called the *exterior derivative* of f.

Normal Bundles

If V is a submanifold of a manifold M, we define the *normal bundle* N_V^M of V in M. It is a vector bundle over V: the quotient of the restriction of TM to V by TV. In other words, for $x \in V$, we have $(N_V^M)_x = (T_xM)/(T_xV)$.

If we choose a *riemannian structure* on M, i.e. a euclidean structure on every T_xM smoothly depending on x, we may identify $(N_V^M)_x$ to the orthogonal complement of T_xV in T_xM.

Tubular Neighborhood

Let V be a submanifold of a manifold M. Note that, for every $x \in V$, we can write $T_xM = T_xV \oplus T_x(N_V^M)_x$—and $T_x(N_V^M)_x$ identifies with $(N_V^M)_x$. One can construct a smooth map $\varphi : N_V^M \to M$ satisfying the following two conditions for every $x \in V$

1. $\varphi(x,0) = 0$;
2. for every $\xi \in (N_V^M)_x = T_x(N_V^M)_x \subset T_xM$, the image of $d\varphi_{(x,0)}(\xi) \in T_x(M)$ in the quotient $(N_V^M)_x = T_xM/T_xV$ is ξ.

By construction, $d\varphi_{(x,0)}$ is an isomorphism for every $x \in V$. It follows, using the inverse function theorem—together with compactness arguments, that there is a neighborhood W of $\{(x,0);\ x \in V\}$ in N_V^M such that φ induces a diffeomorphism from W to a neighborhood of V in M. This construction is called the *tubular neighborhood* construction.

Some Words on Transversality

We finish with just a few words on transversality.

Definition 5 Let M, N be smooth manifolds let $f : M \to N$ be a smooth map.

1. Let V be a submanifold of N. We say that f is *transverse* to V if for every $x \in M$ such that $f(x) \in V$, we have $df_x(T_xM) + T_{f(x)}V = T_{f(x)}N$.
2. Let P be a smooth manifold $g : P \to N$ be a smooth map. We say that f and g are *transverse* if for every $(x, y) \in M \times P$ such that $f(x) = g(x)$, we have $df_x(T_xM) + dg_y(T_yP) = T_{f(x)}N$.

If f is transverse to a manifold V, then $f^{-1}(V)$ is a smooth submanifold of M; if $f : M \to N$ and $g : P \to N$ are transverse, the fibered product $\{(x, y) \in M \times P;\ f(x) = g(x)\}$ is a smooth manifold (it is a smooth submanifold of $M \times P$).

2.2 C^*-Algebras

We now describe a very central object of our study: C^*-algebras. These are "non commutative analogues" of (locally) compact spaces.

We want to study quite singular spaces. A typical example of such a singular space is a quotient space of an equivalence relation, and in particular, the one that we will be concerned with is the space of leaves of a foliation. A (locally) compact space is exactly described by the C^*-algebra of continuous complex valued functions on it (vanishing at infinity)—and every commutative C^*-algebra is of that form (see Theorem 1.1 below).

In this section we give the definitions and some important properties of C^*-algebras. We describe some simple examples. More examples will be given in the next sections, where we will explain how to naturally associate a noncommutative C^*-algebra to a singular space—using *groupoids.*

2.2.1 Definitions and Examples

Let us start with some definitions.

- Let A be a complex algebra. An *involution* of A is a map $a \mapsto a^*$ satisfying:

$$\forall (a,b) \in A^*,\ \forall \lambda \in \mathbb{C},\quad (a+b)^* = a^* + b^*,\ \ (\lambda a)^* = \overline{\lambda} a^*,\ \ (ab)^* = b^* a^*,\ \ (a^*)^* = a.$$

  Note that $1^* = 1$.
- A Banach $*$-algebra is a Banach algebra endowed with an involution $a \mapsto a^*$ such that $\|a^*\| = \|a\|$ (for all $a \in A$). We will assume that the norm of A satisfies $\|ab\| \leq \|a\|\|b\|$ (for all $a, b \in A$).

Definition 6 A C^*-algebra is a Banach $*$-algebra A such that, for every $x \in A$, we have $\|x^* x\| = \|x\|^2$.

(Remark: it follows that $\|1\| = 1$).

Special elements Let A be a unital C^*-algebra and $a \in A$. We say that a is

- *self-adjoint* if $a^* = a$;
- *unitary* if $a^* a = a a^* = 1$;
- *normal* if $a^* a = a a^*$.

Examples of C^*-Algebras

1. Let X be a compact space; denote by $C(X)$ the algebra of continuous functions. We put $f^* = \overline{f}$ (involutive algebra), and $\|f\| = \|f\|_\infty$. It is a C^*-algebra.
2. Let H be a (finite or infinite dimensional) Hilbert space. Denote by $\mathcal{L}(H)$ the algebra of bounded (continuous) linear maps from H to H. Let $\|\ \|$ be the operator norm ($\|T\| = \sup\{\|Tx\|;\ x \in H,\ \|x\| \leq 1\}$ and T^* the usual adjoint defined by the equality $\langle x|T^* y\rangle = \langle Tx|y\rangle$ for all $x, y \in H$). It is a C^*-algebra.
3. A closed $*$-subalgebra of a C^*-algebra is a C^*-algebra. The ideal of compact operators in a Hilbert space is a C^*-algebra.

The next theorem, that we will prove in the next sections, tells us that our examples are in fact the most general examples of C^*-algebras.

Theorem 1 *1. Every unital commutative C^*-algebra is isomorphic to a $C(X)$.*
2. Every C^*-algebra embeds isometrically as a $*$-subalgebra of an $\mathcal{L}(H)$.

The first statement uses two notions of *spectrum*; the second one uses the notion of *positivity* in a C^*-algebra. We will examine these two notions.

2.2.2 Spectrum

In this section we explain how to prove the first statement of Theorem 1. This is based on two notions of spectrum.

Spectrum of an Element in a Banach Algebra, in a C^*-Algebra

We start the following easy proposition.

Proposition 2 *Let A be a Banach algebra and $x \in A$.*

1. *If $\|x\| < 1$, then the series $\sum x^k$ converges in A its $\sum_{k=0}^{+\infty} x^k$ is the inverse of $1 - x$.*
2. *The set A^{-1} of invertible elements of A is open and the map $\kappa : x \mapsto x^{-1}$ is differentiable. We have $d\kappa_x(h) = -x^{-1}hx^{-1}$.*

Proof 1. We have $\|x^k\| \leq \|x\|^k$ and the series $\sum \|x\|^k$ converges since $\|x\| < 1$. It follows that $\sum \|x^k\|$ converges, and since A is complete, $\sum x^k$ converges. We have

$$x\Big(\sum_{k=0}^{+\infty} x^k\Big) = \Big(\sum_{k=0}^{+\infty} x^k\Big)x = \Big(\sum_{k=0}^{+\infty} x^k\Big) - 1.$$

2. If $x \in A^{-1}$ and $\|h\| < \|x^{-1}\|^{-1}$, we have $x + h = x(1 + (x^{-1}h))$. As $\| - x^{-1}h\| < 1$, it follows from 1 that $(1 + (x^{-1}h))$ is invertible. We find $(x + h)^{-1} = \Big(\sum_{k=0}^{+\infty}(-x^{-1}h)^k\Big)x^{-1}$. In particular,

$$(x + h)^{-1} = x^{-1} - x^{-1}hx^{-1} + O(\|h\|^2).$$

The conclusion follows.

□

Definition 7 Let A be a complex Banach algebra and $x \in A$, the *spectrum* of x (relative to A) is the subset $\mathrm{Sp}_A(x) = \{\lambda \in \mathbb{C};\ x - \lambda 1 \text{ not invertible}\}$ of $\mathbb{C}$.

Proposition 3 *Let A be a Banach algebra and $x \in A$. The set $Sp_A(x)$ is a compact subspace of $\mathbb{C}$ and the map $\lambda \mapsto (x - \lambda 1)^{-1}$ from $\mathbb{C} \setminus Sp_A(x)$ to A is holomorphic.*

Proof It follows from Proposition 2 that $\mathrm{Sp}_A(x)$ is closed in $\mathbb{C}$ and that if $|\lambda| > \|x\|$, then $x - \lambda 1 = -\lambda(1 - \lambda^{-1}x)$ is invertible. Therefore $\mathrm{Sp}_A(x)$ is bounded in $\mathbb{C}$, whence compact.

Again by Proposition 2, the map $\lambda \mapsto (x - \lambda 1)^{-1}$ is holomorphic: its derivative at λ is $(x - \lambda 1)^{-2}$. □

The following theorem is proved using holomorphic functions. We skip its proof.

Theorem 2 *Let A be a unital Banach algebra ($A \neq \{0\}$). For $x \in A$, the set $Sp_A(x)$ is a non empty compact subspace of $\mathbb{C}$. We have* $\sup\{|\lambda|;\ \lambda \in Sp(x)\} = \lim \|x^n\|^{1/n}$.

Definition 8 The number $\sup\{|\lambda|;\ \lambda \in \mathrm{Sp}(x)\} = \lim \|x^n\|^{1/n}$ is called the *spectral radius* of x and is denoted by $\rho(x)$.

We will use in the sequel the following theorem.

Theorem 3 (Gelfand–Mazur theorem) *Let A be a Banach algebra with unit* 1. *If A is a division ring, then the map $\lambda \mapsto \lambda 1$ is an isomorphism $j : \mathbb{C} \to A$.*

Proof We prove that j is surjective. Let $x \in A$; since $\mathrm{Sp}_A x \neq \emptyset$, there exists $\lambda \in \mathbb{C}$ such that $j(\lambda) - x$ is not invertible; since A is a division ring, $j(\lambda) = x$. □

Now we examine the case of a C^*-algebra and its "special" elements.

Proposition 4 *Let A be a unital C^*-algebra and $a \in A$;*

1. *if a is unitary, then $Sp_A(a) \subset \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$;*
2. *if a is self adjoint, then $Sp_A(a) \subset \mathbb{R}$;*
3. *if a is normal, then $\rho(a) = \|a\|$.*

Proof 1. If a is unitary, then $\|a\|^2 = \|a^*a\| = 1$. If follows from Proposition 2 that,

- for $|\lambda| > 1$, the element $a - \lambda 1 = -\lambda(1 - \lambda^{-1}a)$ is invertible
- for $|\lambda| < 1$, the element $a - \lambda 1 = a(1 - \lambda a^*)$ is invertible.

2. Let $t \in \mathbb{R}$ with $t > \|a\|$. Then $a \pm it1$ is invertible and $b = (a + it1)^{-1}(a - it1)$ is easily seen to be unitary. It follows that

$$\mathrm{Sp}_A(a) = \{\lambda \in \mathbb{C} \setminus \{-it\};\ \frac{\lambda - it}{\lambda + it} \in \mathrm{Sp}_A(b)\} \subset \{\lambda \in \mathbb{C} \setminus \{-it\};\ \left|\frac{\lambda - it}{\lambda + it}\right| = 1\} = \mathbb{R}.$$

3. For every self adjoint element b, it follows from the definition of C^*-algebras that $\|b^2\| = \|b\|^2$. As $(a^*a)^k$ is self adjoint we find by induction on n that $\|(a^*a)^{2^k}\| = \|a^*a\|^{2^k}$. But, as a is normal $(a^*a)^{2^k} = (a^*)^{2^k}a^{2^k}$ and it follows that $\|a\|^{2^{k+1}} = \|a^*a\|^{2^k} = \|a^{2^k}\|^2$. Finally, $\|a\| = \|a^{2^k}\|^{2^{-k}}$. Taking the limit when $k \to \infty$ we find $\rho(a) = \|a\|$.

□

Exercise 1 Let A be a *real* Banach algebra which is a division ring. Show that A is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$ or to $\mathbb{H}$ (the algebra of quaternions).

Spectrum of a Commutative Banach Algebra

Definition 9 Let A be a Banach algebra. A *character* on A is a nonzero algebra morphism $\chi : A \to \mathbb{C}$. In other words, χ is a linear map such that, for all $a, b \in A$, we have $\chi(ab) = \chi(a)\chi(b)$.

The *spectrum* of a commutative Banach algebra A is the set $\mathrm{Sp}(A)$ of its characters.

The set $\mathrm{Sp}(A)$ is endowed with the *weak topology,* i.e. the weakest topology for which $\chi \mapsto \chi(x)$ is continuous for every x.

Theorem 4 *For $a \in A$; we have $Sp_A(a) = \{\chi(a);\ \chi \in Sp(A)\}$.*

Proof If χ is a character, then $a - \chi(a)1$ is in the kernel of χ, which is an ideal different from A—and it follows that $a - \chi(a)1$ is not invertible, i.e. $\chi(a) \in \mathrm{Sp}_A(a)$.

Conversely, if $\lambda \in \mathrm{Sp}_A(a)$, then the ideal $(a - \lambda 1)A$ is contained in a maximal ideal J (Krull's theorem). As J has an empty intersection with the open set A^{-1} of A, it follows that its closure $\overline{J}$ satisfies $\overline{J} \cap A^{-1} = \emptyset$, whence $\overline{J} \neq A$. By maximality, J is closed. Then A/J is a Banach algebra and a division ring, thus $A/J = \mathbb{C}$ and the quotient map $\chi : A \to A/J$ is a character. As $a - \lambda 1 \in J = \ker \chi$, it follows that $\chi(a) = \lambda$. □

Corollary 1 *The spectrum of a commutative Banach algebra A is contained in the unit ball of the (continuous) dual A^* of A.*

Proof Indeed, for every $\chi \in Sp(a)$ and $a \in A$, we have $\chi(a) \in \mathrm{Sp}_A(a)$, thus $|\chi(a)| \leq \|a\|$, and $\|\chi\| \leq 1$. □

Theorem 5 *The spectrum of a unital commutative Banach algebra, endowed with the weak topology is compact.*

Proof Denote by B the unit ball of A^*. Endowed with the weak topology, B is compact (Banach–Alaoglu theorem, cf. [24]). The spectrum A is the intersection over $a, b \in A$ of the closed subsets $\{\chi \in B;\ \chi(ab) = \chi(a)\chi(b)\}$ of B. □

The *Gelfand transform* is the "obvious" map $G_A : A \to C(\mathrm{Sp}(A))$: we define $G_A(a) : \mathrm{Sp}(A) \to \mathbb{C}$ to be the map $\chi \mapsto \chi(a)$.

The Case of Unital Commutative C^*-Algebras

Let us prove part 1 of Theorem 1. Let us give a more precise statement.

Theorem 6 *If A is a unital commutative C^*-algebra, then G_A is an isomorphism of C^*-algebras.*

Proof Every $x \in A$ is normal; we therefore have $\|x\| = \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}_A(x)\}$. By Theorem 4, it follows that $\|x\| = \sup\{|\chi(x)|;\ \chi \in \mathrm{Sp}(A)\} = \|G_A(x)\|_\infty$ (by definition of G_A).

If $x \in A$ is self-adjoint, it follows from Proposition 4 that for every $\chi \in \mathrm{Sp}(A)$, we have $G_A(x)(\chi) = \chi(x) \in \mathbb{R}$, i.e. $G_A(x)$ is a real function on $\mathrm{Sp}(A)$. It follows that, for every $y \in A$, since $a = \dfrac{y + y^*}{2}$ and $b = \dfrac{y - y^*}{2i}$ are self adjoint, we find $G_A(a) - iG_A(b) = \overline{G_A(a) + iG_A(b)}$, i.e. $G(y^*) = \overline{G(y)}$.

We have proved that G_A is an isometric $*$-preserving morphism. We just need to show that G_A is surjective. This follows from the Stone-Weierstrass theorem that we recall now:

Theorem 7 (Stone-Weierstrass Theorem) *Let X be a compact Hausdorff space. Every unital closed $*$-subalgebra B of the C^*-algebra $C(X)$ which separates points of X is equal to $C(X)$.*

The condition B separates the points means that for every two points $x, y \in X$, if $x \neq y$, there exists $f \in B$, such that $f(x) \neq f(y)$.

In our case, two characters that coincide on A are equal! Therefore $G_A(A)$ separates points. As $G_A(1)$ is the function 1, it follows that $G_A(A)$ is a unital $*$-subalgebra of $C(\mathrm{Sp}(A))$. Finally, since G_A is isometric, $G_A(A)$ is complete and therefore closed in $C(\mathrm{Sp}(A))$. □

Theorem 8 (Continuous functional calculus) *Let a be a normal element in a C^*-algebra A. There exists a (unique) $*$-morphism $\pi_a : C(Sp_A(a)) \to A$ such that $\pi_a(z) = a$ and $\pi_a(1) = 1$, where $z : Sp_A(a) \to \mathbb{C}$ is the map $\lambda \mapsto \lambda$.*

Proof Consider the closed $*$-subalgebra B of A generated by 1 and a. Since a is normal, B is commutative. Given two characters of $\chi, \chi' \in \mathrm{Sp}(B)$, the set $\{x \in B,\ \chi(x) = \chi'(x)\}$ is a closed unital $*$-subalgebra of B. It follows that if $\chi(a) = \chi'(a)$, then $\chi = \chi'$. Therefore the map $\chi \mapsto \chi(a)$ is injective: it is a homeomorphism $p : \mathrm{Sp}(B) \to \mathrm{Sp}_A(a)$. We then have natural isomorphisms $C(\mathrm{Sp}_A(a)) \overset{p^*}{\simeq} C(\mathrm{Sp}(B)) \overset{G_B^{-1}}{\simeq} B$. The morphism π_a is the composition $C(\mathrm{Sp}_A(a)) \xrightarrow{p^*} C(\mathrm{Sp}(B)) \xrightarrow{G_B^{-1}} B \xrightarrow{\subset} A$.

Uniqueness follows from the Stone-Weierstrass Theorem: the C^*-subalgebra of $C(\mathrm{Sp}_A(a))$ generated by 1 and z is equal to $C(\mathrm{Sp}_A(a))$; two $*$-morphisms $C(\mathrm{Sp}_A(a)) \to A$ which coincide on 1 and z are equal. □

If a is normal and $f \in \mathrm{Sp}_A(a)$, the element $\pi_a(f)$ is denoted by $f(a)$.

2.2.3 Positivity—Explanations for Part 2 of Theorem 1

Theorem 9 *Let A be a unital C^*-algebra*

1. Positive elements in a C^*-algebra. *For $x \in A$, the following are equivalent*

   (i) $x = x^*$ *and* $Sp(x) \subset \mathbb{R}_+$.
   (ii) there exists $y \in A$ *such that* $y = y^*$ *and* $x = y^2$
   (iii) there exists $y \in A$ *such that* $x = y^*y$.

2. The positive cone. *The set A_+ of those elements satisfies*

   (a) *for* $x \in A_+$ *and* $\lambda \in \mathbb{R}_+$ *we have* $\lambda x \in A_+$;
   (b) *for* $x, y \in A_+$ *we have* $x + y \in A_+$;
   (c) *for* $x \in A_+$, *if* $-x \in A_+$ *then* $x = 0$.

   We write $x \geq 0$ *for* $x \in A_+$.

Proof 1. (ii)⇒(iii) is obvious.
(i)⇒(ii). If $\mathrm{Sp}_A(x) \subset \mathbb{R}_+$, let $f : \mathrm{Sp}_A(x) \to \mathbb{C}$ be the function $t \mapsto \sqrt{t}$. Since $f = \overline{f}$, $f(x)$ is a self-adjoint element of A and we have $f(x)^2 = x$.

Although this is not really needed, let us prove
(ii)⇒(i). Let $y = y^*$. If $\lambda < 0$ write $\lambda = q^2$ $q \in \mathbb{C}$ imaginary. Then $y - q$ and $y + q$ are invertible since the $\mathrm{Sp}_A(y) \subset \mathbb{R}$. Therefore $y^2 - \lambda = (y - q)(y + q)$ is invertible, and $\mathrm{Sp}_A(y^2) \subset \mathbb{R}_+$.

2. Before proving (iii)⇒(i) let us establish 2, i.e. that the set A_+ of elements satisfying the equivalent conditions (i) and (ii) satisfy conditions (a), (b) and (c).

   - Of course, if $x \in A_+$ and $\lambda \in \mathbb{R}_+$, then $\lambda x \in A_+$.
   - Also, if $x \in A_+$ and $-x \in A_+$, then the spectrum of x is reduced to 0, and therefore its norm, which is equal to its spectral radius (Proposition 4), is 0 too.
   - To establish (b), let us first characterize the elements of A_+: for x self-adjoint we have $x \in A_+ \iff \exists t \in \mathbb{R}_+;\ \|t - x\| \leq t$.
     Indeed $\|t - x\| \leq t \iff \mathrm{Sp}x \subset [0, 2t]$.
     If $x, y \in A_+$, writing $\|t - x\| \leq t$ and $\|s - y\| \leq s$, with $s, t \in \mathbb{R}_+$, it follows that $\|(s + t) - (x + y)\| \leq s + t$.

   Finally let us prove (iii)⇒(i) in statement 1.
   Let $x \in A$. Note that $xP(x^*x) = P(xx^*)x$ for every polynomial P and therefore for every continuous function $f : \mathbb{R} \to \mathbb{R}$.
   Let $f : \mathbb{R} \to \mathbb{R}$ be the function $t \mapsto 0$ if $t \geq 0$ and $t \mapsto t$ if $t \leq 0$. Note that, for every $t \in \mathbb{R}$ we have $f(t)^2 t = f(t)^3 \leq 0$. Put $y = xf(x^*x)$. We have $y^*y = f(x^*x)^3$ and $yy^* = f(xx^*)xx^*f(xx^*) = f(yy^*)^3$; therefore $-y^*y \in A_+$ and $-yy^* \in A_+$.
   On the other hand, writing $y = a + ib$ with a, b self-adjoint, we find $y^*y + yy^* = 2(a^2 + b^2)$. It follows that $y^*y = -yy^* + 2(a^2 + b^2) \in A_+ \cap -A_+$ and therefore

$y^*y = 0$. Finally, $f(t)^3 = 0$ for every $t \in \operatorname{Sp} x^*x$ and therefore $\operatorname{Sp} x^*x \subset \mathbb{R}_+$.

□

Thanks to this theorem, we can now construct morphisms from our C^*-algebra to $\mathcal{L}(H)$ and thus embed A to $\mathcal{L}(H)$.

The GNS construction (GNS stands for Gelfand–Naimark–Segal).

- Let A be a C^*-algebra. A linear map $f : A \to \mathbb{C}$ is said to be *positive* if $f(x) \in \mathbb{R}_+$ for $x \in A_+$.
- Given a positive linear map $f : A \to \mathbb{C}$, define a sesquilinear map $(a, b) \mapsto f(a^*b)$ on A.
- We have a Schwarz inequality: $|f(a^*b)|^2 \leq f(a^*a)f(b^*b)$. It follows that if $f(a^*a) = 0$, then $f(a^*b) = 0$ for every $b \in A$.
- There is a Hilbert space H_f and a linear map $\eta_f : A \to H_f$ such that, for every $a, b \in A$ we have $\langle \eta_f(a) | \eta_f(b) \rangle = f(a^*b)$ and $\eta_f(A)$ is dense in H_f. The kernel of η_f is $N_f = \{x \in A;\ f(a^*a) = 0\}$.
  The space H_f is a *completion* of the quotient space A/N_f.
- There is a morphism $\pi_f : A \to \mathcal{L}(H_f)$ satisfying $\pi_f(a)\big(\eta_f(b)\big) = \eta_f(ab)$.

To conclude the proof of part 2 of Theorem 1, we just construct, thanks to Hahn-Banach theorem, enough positive linear maps on A.

Enveloping C^*-algebras Let $\mathcal{A}$ be a $*$-algebra. Let $(\pi_i)_{i \in I}$ be a family of $*$-representations, i.e. $*$-preserving morphisms $\pi_i : \mathcal{A} \to \mathcal{L}(H_i)$—where H_i is a Hilbert space, for every $i \in I$.

We then construct a C^*-algebra $C^*(\mathcal{A}, (\pi_i)_{i \in I})$ by completing the algebra $\{a \in \mathcal{A};\ \sup_i \|\pi_i(a)\| < +\infty\}$ with respect to the semi-norm $a \mapsto \sup_i \|\pi_i(a)\|$.

If $\mathcal{A}$ is a Banach $*$-algebra, for every $a \in \mathcal{A}$ and $i \in I$, we have $\|\pi_i(a)\|^2 = \|\pi_i(a^*a)\|$ which is equal to the spectral radius of $\pi(a^*a)$. Now, for $|\lambda| > \|a^*a\|$, the element $\lambda - a^*a$ of $\mathcal{A}$ is invertible, and thus $\lambda - \pi_i(a^*a)$ is invertible in $\mathcal{L}(H_i)$. It follows that $\|\pi_i(a)\| \leq \|a\|$ and therefore $\{a \in \mathcal{A};\ \sup_i \|\pi_i(a)\| < +\infty\} = \mathcal{A}$.

Some Facts

1. Let A be a Banach algebra and let $a, b \in A$.
   a. If A is commutative, $\operatorname{Sp}_A(a + b) \subset \operatorname{Sp}_A(a) + \operatorname{Sp}_A(b)$.
   b. Matrices $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ give a counter example in the noncommutative case.
   c. If $ab = ba$, we still have $\operatorname{Sp}_A(a + b) \subset \operatorname{Sp}_A(a) + \operatorname{Sp}_A(b)$. (One can construct a commutative subalgebra B of A containing a, b and such that $\operatorname{Sp}_A(a) = \operatorname{Sp}_B(a)$ and $\operatorname{Sp}_A(a) = \operatorname{Sp}_B(a)$—note then that $\operatorname{Sp}_A(a + b) \subset \operatorname{Sp}_B(a + b) \subset \operatorname{Sp}_B(a) + \operatorname{Sp}_B(b) = \operatorname{Sp}_A(a) + \operatorname{Sp}_A(b)$).

2. Let A be a C^*-algebra and $a \in A$. Then a is unitary (*resp.* self-adjoint) if and only if a is normal and $\mathrm{Sp}_A(a) \subset \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$ (*resp.* $\mathrm{Sp}_A(a) \subset \mathbb{R}$).

3. *Unitarisation of a Banach algebra. Of a C^*-algebra.*

   a. Let $\mathcal{A}$ be a Banach algebra without unit. We endow the vector space $\mathbb{C} \times \mathcal{A}$ with the product $(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab)$. Then $\mathbb{C} \times \mathcal{A}$ is a unital Banach algebra for a natural norm.
   b. Assume A is a C^*-algebra, and put $(\lambda, a)^* = (\bar{\lambda}, a^*)$. Then $\mathbb{C} \times \mathcal{A}$ is a C^*-algebra—and its norm is given by $\|(\lambda, a)\| = \sup\{\|\lambda b + ab\|;\ b \in A,\ \|b\| \leq 1\}$.

   Note also that the map $\epsilon : (\lambda, a) \mapsto \lambda$ is a morphism (with kernel A). This morphism is split by the morphism $\lambda \mapsto (\lambda, 0)$.

4. If B is a closed subalgebra of a Banach algebra A and $b \in B$, we have $\mathrm{Sp}_A(b) \subset \mathrm{Sp}_B(b)$—and in general this is not an equality. We always have $\mathrm{Sp}_A(a) = \mathrm{Sp}_B(A)$ if B is a C^*-algebra.
   We actually used implicitly this fact in the proof of theorem 8.

5. Let $\pi : A \to B$ be a morphism of C^*-algebras. Using functional calculus one can show that

   a. for every $x \in A$, $\|\pi(x)\| \leq \|x\|$;
   b. for every $x \in \pi(A)$, there exists $y \in A$ such that $\|x\| = \|y\|$ and $\pi(y) = x$. In particular, if π is injective, $\|\pi(x)\| = \|x\|$.
   c. It follows that $\pi(A)$ is closed in B.

6. Let J be a closed two sided ideal in a C^*-algebra A, then

   a. J is stable by $x \mapsto x^*$;
   b. endowed with the quotient norm, A/J is a C^*-algebra.

2.3 *K-Theory of Operator Algebras*

By what we saw (Theorem 1.1) the C^*-algebras are non commutative generalizations of compact spaces. We will use them in order to study *non commutative* (singular) spaces.

It is then natural to try and extend constructions of compact spaces to the non commutative case. What invariants of compact spaces can be extended to the non commutative world? The easiest one is K-theory.

In this section, we will give the definitions of K-theory and we will outline the main properties of the K-theory of C^*-algebras.

2.3.1 Definition of K-Theory of a C*-Algebra

K^0 of a Space

Let X be a compact space. The K-theory classifies vector bundles. We will only talk here about the complex K-theory, i.e. the classification of complex vector bundles. One forms a group: it is the group of formal differences of vector bundles. In the same way we define $\mathbb{Z}$ out of $\mathbb{N}$, we define a group $K^0(X)$ to be the set of formal differences of finite dimensional complex vector bundles.

Definition 10 Let X be a compact space. The group $K^0(X)$ is the group of formal differences $[E]-[F]$ of finite dimensional complex vector bundles. We set $[E_1]-[F_1]=[E_2]-[F_2]$ if there exists a finite dimensional complex bundle G such that the complex vector bundles $E_1\oplus F_2\oplus G$ and $E_2\oplus F_1\oplus G$ are isomorphic.

The reason why the bundle G appears here is that the set of vector bundles does not always satisfy the *cancelation property*, i.e. $E_1\oplus F$ may be isomorphic to $E_2\oplus F$ though E_1 and E_2 are not isomorphic.

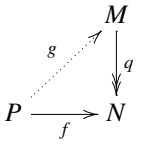
The group $K^0(X)$ is abelian and its composition law is denoted by $+$. The sum in $K^0(X)$ is defined by $([E_1]-[F_1])+([E_2]-[F_2])=[E_1\oplus E_2]-[F_1\oplus F_2]$.

K_0 of a Ring-of a C^*-Algebra

Theorem 10 (Serre–Swan Theorem) *Let X be a compact space.*

1. *Let E be a finite dimensional complex vector bundle over X. The space $\Gamma(E)=C(X;E)$ of continuous sections of E is a* finitely generated projective module *over the algebra $C(X)$.*
2. *Conversely, for every finitely generated projective module $\mathcal{E}$ over $C(X)$, there is a finite dimensional complex vector bundle E over X such that the module $\mathcal{E}$ and $C(X;E)$ are isomorphic.*
3. *Given two finite dimensional complex bundles E and F, the modules $C(X;E)$ and $C(X;F)$ are isomorphic if and only if the bundles E and F are isomorphic.*

Let us give a few words of explanation:
A (say right) module P over a ring A is said to be *projective* if every morphism in a quotient module lifts. This is expressed in the diagram



If P is also *finitely generated*, we have a surjective module morphism $q:A^n\to P$. Lifting the identity of P gives a module morphism $g:P\to A^n$ such that $q\circ g$ is the identity of P. Then $p=g\circ q$ is an idempotent morphism of A^n. Now, (right) module morphisms of A^n are just $n\times n$ matrices over A. In other words finitely generated

projective right modules over A are of the form pA^n where p is an idempotent of the ring $M_n(A)$.

Definition 11 If A is a ring, we denote by $K_0(A)$ the group of formal differences of finitely generated projective (right) A-modules.

By the Serre–Swan theorem, we find $K^0(X) = K_0(C(X))$ for every compact space X.

Remark 3 The group $K^0(X)$ is a ring. The multiplication comes from the tensor product of vector bundles. This is no longer true in the case of non commutative C^*-algebras: the tensor product of two right modules is not well defined.

Example A finitely generated module over a field K is just a finite dimensional vector space. Those are just classified by their dimension. It follows that $K_0(K) = \mathbb{Z}$.

2.3.2 Properties

K_0 as a "functor". If $f : A \to B$ is a ring morphism, then, to every finitely generated projective A-module E, we associate an *induced* B-module $E \otimes_A B$. The easiest way to understand this module is by saying that if $E = pA^n$ with $p \in M_n(A)$ an idempotent, then $E \otimes_A B = f(p)B^n$ where $f(p)$ is the associated idempotent in $M_n(B)$—we have extended f to $n \times n$ matrices by setting $f\big((a_{i,j})\big) = \big(f(a_{i,j})\big)$.

In this way we obtain a morphism $f_* : K_0(A) \to K_0(B)$.

Homotopy invariance. If A and B are Banach algebras, we define a homotopy of morphisms from A to B to be a family (h_t) of morphisms from A to B such that, for all $a \in A$, the map $t \mapsto h_t(a)$ is continuous (from $[0, 1]$ to B). In other words h is a morphism from A to the algebra $C([0, 1]; B)$ of continuous functions from $[0, 1]$ to B.

If p and q are idempotents in some Banach algebra such that $\|p - q\|$ is small, then $p + q - 1$ is near $2p - 1$ which is invertible—we have $(2p - 1)^2 = 1$. More precisely, if $\|p - q\|\|2p - 1\| < 1$, then

$$(p + q - 1)(2p - 1) = (2p - 1)^2 - (p - q)(2p - 1) = 1 - (p - q)(2p - 1)$$

is invertible; its inverse is the sum $\sum_{n=0}^{+\infty} \big((p - q)(2p - 1)\big)^n$. It follows that $p + q - 1$ is invertible. But $p(p + q - 1) = pq = (p + q - 1)q$, whence $q = (p + q - 1)p(p + q - 1)^{-1}$ and the elements p and q define the same element in K_0.

It follows that if $h_0 : A \to B$ and $h_1 : A \to B$ are homotopic, then they define the same morphism $K_0(A) \to K_0(B)$. In other words, $(h_0)_* = (h_1)_*$.

Non unital algebras. Many of the C^*-algebras that we encounter are not unital. If $\mathcal{A}$ is a non unital algebra, there is a unital algebra $\widetilde{\mathcal{A}}$ and a split short exact sequence

$$0 \to \mathcal{A} \longrightarrow \widetilde{\mathcal{A}} \xrightarrow{\epsilon} \mathbb{C} \to 0$$

(see Fact 3).

Definition 12 If A is a non-unital complex algebra, the group $K_0(A)$ is the kernel of $\epsilon_* : K_0(\widetilde{A}) \to K_0(\mathbb{C})$. It is isomorphic to the quotient $K_0(\widetilde{A})/K_0(\mathbb{C})$ (where $K_0(\mathbb{C})$ embeds in $K_0(\widetilde{A})$ via the unital morphism $\mathbb{C} \to \widetilde{A}$).

Remark: K-theory with compact supports. Let X be a locally compact (non compact) space. The group $K^0(X)$ can be described using K-theory with compact supports.

- Cycles for this K theory are triples (E^+, E^-, σ) where $E^\pm$ are vector bundles over X and σ is a morphism of vector bundles, i.e. a continuous family $\sigma_x : E^+_x \to E^-_x$ (for $x \in X$ such that σ_x is invertible outside a compact subset of X.
- Homotopic triples define the same element, and if σ is invertible everywhere, the triple (E^+, E^-, σ) is the zero element.
- The addition is the obvious one.

One defines in this way a group which coincides with $K^0(X) = K_0(C_0(X))$.

This kind of definitions generalizes also to the noncommutative case. Let A be a C^*-algebra without unit. Let M be a unital C^*-algebra which contains A as a closed two-sided ideal and let $Q = M/A$ be the quotient C^*-algebra. One defines K-cycles for A as being triples (E^+, E^-, σ) where $E^\pm$ are finitely generated projective (right) modules over M, $\sigma : E^+ \to E^-$ is a an M-linear map such that $\sigma \otimes_M 1 : E^+ \otimes_M Q \to E^- \otimes_M Q$ is an isomorphism.

One defines in this way a group which coincides with $K_0(A)$.

2.3.3 Exact Sequences and K_1

If J is a closed two sided ideal in a Banach algebra A, we have a short exact sequence

$$0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0,$$

one shows easily that the corresponding sequence for the K_0 groups is exact in its middle term, i.e. $j_*(K_0(J)) = \ker(q_*)$. On the other hand, this sequence is not always exact in the other terms: j_* is not injective in general and q_* is not surjective. In particular, if we let CA to be the algebra of continuous functions $[0, 1] \to A$ vanishing at 1 and SA its ideal of functions vanishing at 0 and 1, we obtain an exact sequence $0 \to SA \to CA \to A \to 0$. The algebra CA is *contractible*: the identity $\mathrm{id}_{CA} : CA \to CA$ is homotopic to the 0 morphism. It follows that $K_0(CA) = 0$.

Definition of K_1. For this reason, it is quite natural to define $K_1(A) = K_0(SA)$.

In fact, the group K_1 has another very natural definition: one puts $K_1(A) = \lim_{\to n} \pi_0(GL_n(A))$. Here

- $GL_n(A)$ is the set of invertible elements in $M_n(A)$;
- $\pi_0(GL_n(A))$ is he set of connected components;
- $GL_n(A)$ embeds in $GL_{n+1}(A)$ through the map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$.
- The group law in $K_1(A)$ is given by the group structure of $GL_n(A)$. Note that for x in $GL_n(A)$, the elements $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ are in the same connected component of $GL_{2n}(A)$, and it follows that the group $K_1(A)$ is commutative.

Example Since $GL_n(\mathbb{C})$ is connected for every n, it follows that $K_1(\mathbb{C}) = \{0\}$.

One then may construct an exact sequence $K_1(J) \to K_1(A) \to K_1(A/J) \to K_0(J) \to K_0(A) \to K_0(A/J)$. This sequence can of course be continued to the left by defining $K_2(A) = K_1(SA)$. But in fact the story ends here—for complex Banach algebras: we have the *Bott periodicity.*

Theorem 11 (Bott periodicity) $K_1(SA) = K_0(A)$.

We now have a "six term exact sequence" associated with an exact sequence $0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$ of Banach algebras:

$$\begin{array}{ccccc} K_1(J) & \xrightarrow{j_*} & K_1(A) & \xrightarrow{q_*} & K_1(A/J) \\ \uparrow{\scriptstyle \partial} & & & & \downarrow{\scriptstyle \partial} \\ K_0(A/J) & \xleftarrow[q_*]{} & K_0(A) & \xleftarrow[j_*]{} & K_0(J) \end{array}$$

An example: Fredholm index. Let H be an infinite dimensional Hilbert space. It is not hard to prove that the C^*-algebra $\mathcal{L}(H)$ of all continuous operators (linear maps) from H to H has trivial K-theory: using an isomorphism H with $H \otimes \ell^2(\mathbb{N})$ we have an endomorphism $j_\infty : \mathcal{L}(H) \to \mathcal{L}(H)$ which to an operator T associates the image through this isomorphism of $T \otimes \mathrm{id}_{\ell^2(\mathbb{N})}$. This represents "infinitely many copies of T". Obviously, the map $\mathrm{id} \oplus j_\infty : \mathcal{L}(H) \to M_2(\mathcal{L}(H))$ is conjugate to j_∞ and it follows that $\mathrm{id}_* + (j_\infty)_* = (j_\infty)_*$, whence $\mathrm{id}_* = 0$ and $K_*(\mathcal{L}(H)) = 0$.

In $\mathcal{L}(H)$, we have a closed two-sided ideal $\mathcal{K}(H)$ of *compact operators*, those for which the image of the closed unit ball is compact (for the norm topology of H). Also $\mathcal{K}(H)$ is the norm closure of the set of continuous operators of finite rank. The C^*-algebra $\mathcal{K}(H)$ is therefore a C^*-algebraic inductive limit of matrix algebras $M_n(\mathbb{C})$—with inclusions $M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. It has the same K-theory as $\mathbb{C}$, namely $K_0(\mathcal{K}(H)) = \mathbb{Z}$ and $K_1(\mathcal{K}(H)) = 0$.

The quotient $\mathcal{L}(H)/\mathcal{K}(H)$ of $\mathcal{L}(H)$ by $\mathcal{K}(H)$ is a C^*-algebra called the *Calkin algebra*. We have an isomorphism $\partial : K_1(\mathcal{L}(H)/\mathcal{K}(H)) \to \mathbb{Z}$. This map is the *Fredholm index* defined as follows.

An operator $T \in \mathcal{L}(H)$ which is invertible in the quotient $\mathcal{L}(H)/\mathcal{K}(H)$ has finite dimensional kernel and its image has finite codimension in H, equal to the dimension of the kernel of T^*. We put $\mathrm{ind}(T) = \dim \ker T - \dim \ker T^*$. One then has $\partial([q(T)]) = \mathrm{ind}(T)$ where $q(T)$ denotes the image of T in the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$ and $[q(T)]$ its class in $K_1(\mathcal{L}(H)/\mathcal{K}(H))$.

2.3.4 Further Properties

The K-theory of C^*-algebras has many other nice and useful properties. Let us outline quickly a few of them.

Thom isomorphism. The Bott periodicity tells us that $K(S^2A) = K(A)$. In particular, $K^*(X \times \mathbb{C}) = K^*(X)$ for every locally compact space X, and thus $K^*(X \times \mathbb{C}^n) = K^*(X)$. In fact, this can be generalized in the following way:

Theorem 12 (Thom isomorphism) *Let E be a complex vector bundle over a locally compact space X. Then there is a well defined isomorphism $K^*(E) = K^*(X)$.*

We will describe the Thom isomorphism in Remark 6

We will describe the inverse of the "Thom map" $K^*(E) \to K^*(X)$ in a quite general case—assuming that X is a manifold in Sect. 2.5.3.

Morita equivalence. Let A be a C^*-algebra. Embedding A in some $\mathcal{L}(H)$ for a Hilbert space H, identifies $M_n(A)$ as a C^*-subalgebra of $\mathcal{L}(H^n)$. The closure of the union of $M_n(A)$ in $\mathcal{L}(H \otimes \ell^2(\mathbb{N}))$ through the inclusions $\mathcal{L}(H^n) \to \mathcal{L}(H^{n+1})$ given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$—is a C^*-algebra denoted as $A \otimes \mathcal{K}$.

The inclusion of A in $A \otimes \mathcal{K}$ is an isomorphism in K-theory.

Two C^*-algebras A and B are Morita equivalent if $A \otimes \mathcal{K}$ and $A \otimes \mathcal{K}$ are isomorphic.

Theorem 13 *Morita equivalent C^*-algebras have the same K-groups.*

A few words on Kasparov's KK-theory. Let us just say that G.G. Kasparov constructed a bivariant K-theory for C^*-algebras. This bivariant theory has many very nice properties and is very much suited for index theory. Thanks to Kasparov's "intersection product", Kasparov's KK-theory acts on K-theory. All the morphisms in K-theory constructed here are in fact naturally constructed as KK-elements. This in a way clarifies their construction as well as their properties.

Explanations on KK-theory can be found in [17, 18]. See also [5, 14, 26, 28].

2.4 Lie Groupoids

In mathematics, one encounters several formulas of convolution.

On a Group

- On $\mathbb{R}$: if f and g are say continuous integrable functions on $\mathbb{R}$, we may define their convolution $f * g$ to be the function defined by $f * g(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dt$.
- On $\mathbb{R}/\mathbb{Z}$: if f and g are continuous periodic functions on $\mathbb{R}$ of period 1, we define their convolution $f * g$ to be the function defined by $f * g(x) = \int_a^{a+1} f(t)g(x-t)\,dt$.
- More generally, if G is a locally compact group and f and g are continuous integrable functions on G (with respect to the left Haar measure), we may define their convolution $f * g$ to be defined (almost everywhere) by $f * g(x) = \int_G f(y)g(y^{-1}x)\,dy$.

Convolution of "kernels"

- Let M be a smooth compact manifold. A *kernel* on M is just a smooth function on $M \times M$. If k_1, k_2 are kernels on M, we may define their convolution by setting $k_1 * k_2(x, y) = \int_M k_1(x, z)k_2(z, y)\,dz$.
- This formula reminds of a very familiar one: the product of matrices where we just have a finite sum instead of an integral.

All theses convolution products give rise to (associative) algebras. They are all particular cases of convolutions on groupoids.

Groupoids are natural objects which generalize both spaces and groups. One can naturally associate a groupoid to many singular spaces. Thanks to this convolution, we will be able to associate a C^*-algebra to every (Lie) groupoid and therefore to many "singular manifolds".

2.4.1 Definition of a Lie Groupoid

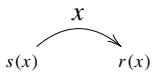
Definition of a Groupoid

Definition 13 A groupoid is a small category such that every arrow is invertible.

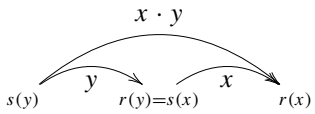
This means that

- We have a set of objects $G^{(0)}$ and a set of arrows $G^{(1)} = G$. (The category is *small*, means that the objects and arrows are not too large: they form sets.)

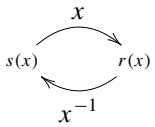
- Every arrow x has a source (or domain) $s(x) \in G^{(0)}$ and a range (or target) $r(x) \in G^{(0)}$.



- The arrows x, y are *composable* if $s(x) = r(y)$ and, in that case, we may compose them and obtain an arrow $x \cdot y$ (or xy) with source $s(y)$ and range $r(x)$.



- *Associativity:* given arrows x, y, z such that $s(x) = r(y)$ and $s(y) = r(z)$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- Every object u has a unit arrow e_u with range and source u; we have $e_{r(x)} \cdot x = x$ and $x \cdot e_{s(x)} = x$ for all $x \in G$; one often identifies $G^{(0)}$ to G through the inclusion $u \mapsto e_u$.
- Finally, every $x \in G$, has an inverse, i.e. there exists x^{-1} in G with $r(x^{-1}) = s(x)$, $s(x^{-1}) = r(x)$, such that $x \cdot x^{-1} = e_{r(x)}$ and $x^{-1} \cdot x = e_{s(x)}$.



We will often write $G \overset{r,s}{\rightrightarrows} M$ or just $G \rightrightarrows M$ to say that G is a groupoid with set of objects $G^{(0)} = M$ (and range and source maps r, s).

Definition of a Lie Groupoid

A Lie groupoid is a groupoid such that G and $G^{(0)}$ are manifolds and all maps (s, r, inclusion $u \mapsto e_u$ of $G^{(0)}$ to G, composition, inverse...) are smooth. Furthermore, the maps r, s are assumed to be submersions; it follows that the fibered product $G^{(2)} = \{(x, y);\ s(x) = r(y)\}$ is a submanifold of $G \times G$—and therefore gives a meaning to the statement: $(x, y) \mapsto x \cdot y$ is smooth.

2.4.2 Examples of Lie Groupoids

1. A smooth manifold M is a Lie groupoid $M \rightrightarrows M$. All the elements are units, the maps s, r, composition are all the identity of M.
2. A Lie group is a Lie groupoid with only one unit. All elements are composable.
3. A vector bundle $p : E \to M$ is a groupoid with objects M. We have $s = r = p$ and the composition of elements $\xi, \eta \in E$ such that $r(\eta) = s(\xi) = x \in M$ is the addition in the fiber E_x (which is a vector space). More generally we could consider fibrations $E \to M$ whose fibers are Lie groups.
4. Given a manifold M, its *pair groupoid* is the groupoid $M \times M$ with objects M. The maps r, s are given by $r(x, y) = x$ and $s(x, y) = y$. The composition is given by $(x, y) \cdot (y, z) = (x, z)$.

5. *Action groupoids.* [Action groupoid] Let a group G act on a space M. We define the groupoid structure on the set $G \times M$, with objects M by setting $s(g, x) = x$, $r(g, x) = g \cdot x$ and the composition $(g, x)(h, y) = (gh, y)$ (when $x = h \cdot y$). This groupoid is called the *action groupoid* and is denoted by $G \ltimes M$ or $M \rtimes G$. If G is a Lie group acting smoothly on a manifold M, the groupoid $G \ltimes M$ is a Lie groupoid.
6. If $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ are Lie groupoids, their direct product is the groupoid $G_1 \times G_2 \rightrightarrows M_1 \times M_2$ defined by $r(x_1, x_2) = (r(x_1), r(x_2))$, $s(x_1, x_2) = (s(x_1), s(x_2))$ and $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2, y_2)$ for x_1, y_1 composable elements of G_1 and x_2, y_2 composable elements of G_2.
7. A sub Lie groupoid of a Lie groupoid G is a submanifold of G which is stable by all the groupoid maps (r, s, inverse, composition). Here are natural examples of subgroupoids.

   - A subgroupoid R of the pair groupoid $M \times M$ is an equivalence relation (on the set $R^{(0)} = \{x \in M;\ (x, x) \in R\}$); for example, given a fibration $\pi : M \to B$, the set $\{(x, y) \in M \times M;\ \pi(x) = \pi(y)\}$ is a Lie-subgroupoid of $M \times M$. Actually, if R is a closed submanifold which is a subgroupoid of $M \times M$, then the quotient space M/R is a (Hausdorff) smooth manifold B and $R = \{(x, y) \in M \times M;\ \pi(x) = \pi(y)\}$.
   - If $G \rightrightarrows M$ is a groupoid, and $A \subset M$ the set $G_A^A = \{g \in G;\ r(g) \in A, s(g) \in (A)\}$ is a groupoid with objects A; if $x \in M$, the set $G_x^x = \{g \in G;\ r(g) = s(g) = x\}$ is a group called the *isotropy* group of x.
   - The action groupoid $M \rtimes G$ is in fact a subgroupoid of the product of the pair groupoid $M \times M$ with the group G: it is the set $\{(x, g, y) \in M \times G \times M;\ x = g \cdot y\}$.
   - Let $G \rightrightarrows M$ be a Lie groupoid. One can define the *s-connected component* of G to be the smallest open subgroupoid of G: it is the union over $x \in M$ of the connected components of x in the manifold $G_x = \{g \in G;\ s(g) = x\}$.

8. Let M be a manifold. The *Poincaré groupoid* or *fondamental groupoid* $\Pi_1(M)$ of M is the set of homotopy classes with fixed ends of smooth paths in M. In other words every smooth path $[0, 1] \to M$ defines an element in $\Pi_1(M)$; two paths α_0 and α_1 define the same element if there exists a smooth map $\alpha : [0, 1] \times [0, 1] \to M$ such that, the maps $t \mapsto \alpha(t, 0)$ and $t \mapsto \alpha(t, 1)$ are constant and for every $t \in [0, 1]$, $\alpha(0, t) = \alpha_0(t)$ and $\alpha(1, t) = \alpha_1(t)$. The isotropy $\Pi_1(M)_x^x$ is the fundamental group $\pi_1(M; x)$ of the (connected component of x in the) manifold M.
9. More generally, let $G \rightrightarrows M$ be an *s-connected* Lie groupoid (i.e. such that the manifolds $G_x = \{g \in G;\ s(g) = x\}$ are connected for every $x \in M$). One can define the *monodromy* groupoid of G to be the set of homotopy classes with fixed end of paths $\gamma : [0, 1] \to G$ such that $\gamma(0)$ is a unit and $t \mapsto s(\gamma(t))$ is constant. It is a Lie groupoid (which may be non Hausdorff).
10. Our constructions on foliations will be based on the *holonomy groupoid* of a foliation. This will be constructed in Sect. 3.2.

11. There are several natural constructions in the literature associated with manifolds with singularities: orbifolds, manifolds with boundaries, corners, manifolds with conical singularities, stratified manifolds. (see [10, 11, 19, 21]; see also [13] and literature there).

Notations for a Lie groupoid We will use the following notation on Lie groupoids. Let $G \overset{r,s}{\rightrightarrows} M$ be a Lie groupoid

- For $u \in M$, we put $G^u = \{x \in G;\ r(x) = u\}$, and $G_u = \{x \in G;\ s(x) = u\}$.
- If A, B are subsets of M, we put $G^A = \{x \in G;\ r(x) \in A\}$, $G_B = \{x \in G;\ s(x) \in B\}$ and $G_B^A = G^A \cap G_B$
- We denote by $G^{(2)}$ the set $G^{(2)} = \{(x, y) \in G \times G;\ s(x) = r(y)\}$ of composable arrows.

2.4.3 Lie Algebroid of a Lie Groupoid

To a Lie group G one associates a *Lie algebra.* This is the tangent space $\mathfrak{G} = T_eG$ at the identity element $e \in G$. One can identify it as the set of left invariant vector fields on G. If X, Y are left invariant vector fields on G, their bracket $[X, Y]$ is also left invariant. In this way, there is a natural Lie bracket on $\mathfrak{G}$: it is antisymmetric ($[Y, X] = -[X, Y]$) and satisfies the *Jacobi identity*: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Now, to a Lie groupoid $G \rightrightarrows M$, one associates its *Lie algebroid* which is a vector bundle over M. In a symmetric way, it is the normal bundle $\mathfrak{A}G = N_M^G$ to the inclusion $M = G^{(0)} \subset G$.

The map $r : G \to M$ is a submersion, and the inclusion $M \to G$ as units is a section of r (and of s). If $x \in M \subset G$ is a unit, then one can write $T_xG = T_xM \oplus T_xG^x$ (where $G^x = \{\gamma \in G;\ r(\gamma) = x\}$). We thus identify $(N_M^G)_x$ with T_xG^x which is the kernel $\ker(dr)_x$ of the differential of $r : G \to M$ at x. Moreover, every element $\gamma \in G$ defines a diffeomorphism $g \mapsto \gamma \cdot g$ form $G^{s(\gamma)}$ to $G^{r(\gamma)}$. As in the Lie group case, we may then identify the space of sections of $\mathfrak{A}G$ with the space of G-invariant sections of the bundle $\ker dr$ of G. Again, if X and Y are smooth G-invariant sections of the bundle $\ker dr$, their commutator $[X, Y]$ is G-invariant. In this way, the space of smooth sections of $\mathfrak{A}G$ is a Lie algebra (of infinite dimension).

Moreover we have an *anchor* which is a vector bundle map $\mathfrak{A}G \to TM$ induced by the map $s : G \to M$ whose differential maps $(\mathfrak{A}G)_x = T_xG^x$ to T_xM.

2.4.4 Deformation Groupoids

The deformation to the normal cone is a very natural geometric construction.[1] In *non commutative geometry*, this construction appeared in the very natural idea of the tangent groupoid constructed by Alain Connes (cf. [8]). This construction allowed him to give a very natural proof of the Atiyah-Singer index theorem that we will outline in the last section of these notes. Since Connes' construction, several generalisations of the tangent groupoid have been used by many authors.

Connes' Tangent Groupoid

Let M be a manifold. Let us assume for simplicity that M is compact. Take a finite cover of M by charts $U_i \subset M$ with diffeomorphisms to $\varphi_i : U_i \to V_i$ where $V_i \subset \mathbb{R}^n$ is an open subset. If x, y are points that are close enough, they live in a same chart. We define the tangent groupoid $G_M \rightrightarrows M \times \mathbb{R}$ of M by setting $G_M = (M \times M) \times \mathbb{R}^* \cup TM \times \{0\}$.

The groupoid structure is given by

- $r(x, y, t) = (x, t)$, $s(x, y, t) = (y, t)$, $(x, y, t)(y, z, t) = (x, z, t)$ for $t \neq 0$;
- $r(x, \xi, 0) = (x, 0) = s(x, \xi, 0)$, $(x, \xi, 0)(y, \eta, 0) = (x, \xi + \eta, 0)$.

The set G_M is naturally equipped with a smooth manifold structure. Let us just describe its topology:

- the set $(M \times M) \times \mathbb{R}^*$ is an open subset, and $TM \times \{0\}$ is a closed subset (with their usual topology).
- a sequence $(x_n, y_n, \lambda_n) \in (M \times M) \times \mathbb{R}^*$ converges to $(x, \xi, 0)$ (with $x \in M, \xi \in T_x M$) if and only if $\lambda_n \to 0$, $x_n \to x$, $y_n \to x$ and $\dfrac{x_n - y_n}{\lambda_n} \to \xi$.

This last equality means that, for every chart U_i around x, which contains x_n and y_n for n large enough, we have $\dfrac{\varphi_i(x_n) - \varphi_i(y_n)}{\lambda_n} \to d\varphi_i(\xi)$.

Equivalently: for any smooth function $f \in C^\infty(M)$, we have

$$\xi(f) = \lim_{n \to \infty} \frac{f(x_n) - f(y_n)}{\lambda_n}\,.$$

Deformation to the Normal Cone

We now describe the smooth structure of Connes' tangent groupoid G_M.

Connes' tangent groupoid is in fact a particular case of a geometric construction called *deformation to the normal cone.* Let V be submanifold of a manifold M. The

[1] Related to the classical notion of blowup that we will not use here.

deformation to the normal cone of V in M is a manifold $DNC(M, V)$ which is the union of $M \times \mathbb{R}^*$ with the (total space of the normal bundle) N_V^M of V in M.

The smooth structure of $DNC(M, V)$ is defined using a tubular neighborhood construction.

The smooth structure of $DNC(M, V)$ satisfies

- the map $j : DNC(M, V) \to M \times \mathbb{R}$ defined by $j(x, y, t) = (x, t)$ for $x \in M$ and $t \neq 0$ and $j(x, \xi, 0) = (x, 0)$ for $x \in V$, $\xi \in (N_V^M)_x$ is smooth;
- for any smooth function $f : M \to \mathbb{R}$ such that $f(x) = 0$ for every $x \in V$, the map $\varphi_f : DNC(M, V) \to \mathbb{R}$ defined by $\varphi_f(x, t) = \dfrac{f(x)}{t}$ for $x \in M$ and $t \neq 0$ and $\varphi_f(x, \xi, 0) = df_x(\xi)$ is smooth[2];
- the maps j and q_f generate the smooth structure of $DNC(M, V)$.[3]

The DNC construction is "functorial": given submanifolds $V_1 \subset M_1$ and $V_2 \subset M_2$ and a smooth map $f : M_1 \to M_2$ such that $f(V_1) \subset V_2$, we obtain a smooth map $DNC(f) : DNC(M_1, V_1) \to DNC(M_2, V_2)$ defined by $DNC(f)(x, t) = (f(x), t)$ for $(x, t) \in M_1 \times \mathbb{R}^*$ and $DNC(f)(x, \xi, 0) = (f(x), df_x(\xi), 0)$ for $x \in V_1, \xi \in (N_{V_1}^{M_1})_x$—note that $df_x : T_xM_1 \to T_{f(x)}M_2$ maps T_xV_1 to $T_{f(x)}V_2$ and therefore defines a map $(N_{V_1}^{M_1})_x \to (N_{V_2}^{M_2})_{f(x)}$ that we still denote df_x.

Remark 4 Given a tubular neighborhood construction $\varphi : N_V^M \to M$ (cf. Sect. 2.1.3), which is a diffeomorphism from N_V^M to an open subset W of M, we may identify $DNC(M, V)$ with the open subset $W \times \mathbb{R} \cup M \times \mathbb{R}^*$ of $M \times \mathbb{R}$.

Deformation Groupoids

From the functoriality of the DNC construction, it follows that if G is a Lie groupoid and $\Gamma \subset G$ is a subgroupoid and a smooth submanifold, then $DNC(G, \Gamma)$ is a Lie groupoid with set of objects $DNC(G^{(0)}, \Gamma^{(0)})$ range and source maps $DNC(r)$ and $DNC(s)$ and composition law $DNC(m)$ where m is the composition law of G.

Connes' tangent groupoid is the particular case when $G = M \times M \rightrightarrows M$ and $\Gamma = G^{(0)} = M \rightrightarrows M$ sitting in $M \times M$ as the diagonal of M. Note indeed that the normal bundle of the diagonal inclusion $M \subset M \times M$ identifies with the tangent bundle of M.

Many other cases appear in the literature. A systematic study of the general construction can be found in [13], where several references are also given.

In [22, 23] is studied the deformation $DNC(G, G^{(0)})$ for any Lie groupoid G. We will only use this deformation groupoid which is very much relevant for index problems.

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[2] Note that since $f(y) = 0$ for $y \in V$, $\xi(f) = 0$ for every $\xi \in T_x(V)$; thus, for $\xi \in T_xM$, $\xi(f)$ only depends on the image of ξ in $(N_V^M)_x$.

[3] This means that functions of this kind can be used to define local coordinates at any point.

2.5 C^*-Algebras of Lie Groupoids

2.5.1 Convolution on a Lie Groupoid

Let G be a Lie groupoid. We can form the convolution of smooth functions f_1, f_2 on G with compact support by setting

$$f_1 * f_2(x) = \int_{x_1, x_2 \in G;\ x_1 x_2 = x} f_1(x_1) f_2(x_2)\, d\nu$$

The set $\{x_1, x_2 \in G;\ x_1 x_2 = x\}$ is a smooth manifold: it can be parametrized by $x_1 \in G^{r(x)} = \{y \in G;\ r(y) = r(x)\}$, and then $x_2 = x_1^{-1} x$. In order to perform correctly this integral, we need to explain this "$d\nu$". It is a smooth "Haar system" i.e. a smooth, left invariant choice of a smooth measure ν^u on every G^u. Left invariance means that, for every $x \in G$, the measure $\nu^{s(x)}$ and $\nu^{r(x)}$ correspond to each other under the diffeomorphism $y \mapsto x \cdot y$ from $G^{s(x)}$ with $G^{r(x)}$.

The convolution formula writes

$$f_1 * f_2(x) = \int_{G^{r(x)}} f_1(y) f_2(y^{-1}x)\, d\nu^{r(x)}(y).$$

This convolution product is associative (using invariance of the Haar system and the Fubini theorem).

The adjoint of $f \in C_c^\infty(G)$ is the function $f^* : x \mapsto \overline{f(x^{-1})}$.

Let us finish this section with two remarks without many explanations—and that we will not use in the text.

Remarks

(a) $(*)$, $[\star\star]$. **Half densities.** There is a more canonical way to define the convolution algebra of a Lie groupoid, avoiding the choice of a smooth Haar system. Instead of taking functions on G, we take sections of a 1-dimensional bundle (which is trivialised by any choice of Haar system). This bundle is a natural bundle of *half densities* over G.

(b) $(*)$, $[\star\star]$ **The non Hausdorff case.** The holonomy groupoid of a foliation—which is the main object of study for us—is not always Hausdorff. There is still a replacement for the algebra $C_c^\infty(G)$ which is made of functions on G that are sums of smooth functions with compact support on open Hausdorff subsets of G (extended by 0 elsewhere). We may note that these functions are *not continuous* as functions on G. But still, they form a $*$-algebra.

2.5.2 The C^*-Algebra of a Lie Groupoid

The C^*-algebra of a groupoid G is a C^*-algebra in which the $*$-algebra $C_c^\infty(G)$ is dense. Since a C^*-algebra is a closed subalgebra in some $\mathcal{L}(H)$, the C^*-algebra of G is the closure of the image of $C_c^\infty(G)$ in some $*$-representation $\pi : C_c^\infty(G) \to \mathcal{L}(H)$.

Unitary Representations of a Groupoid

In order to understand what these representations are let us look at two "easy cases":

1. G is a group;
2. G is just a manifold M.

The group case. Let us start with the case where G is a group, and even a discrete group (a manifold of dimension 0).

The algebra $C_c^\infty(G)$ is then spanned by characteristic functions of group elements δ_x (with $\delta_x(y) = 1$ if $y = x$ and $\partial_x(y) = 0$ if $x \neq y$).

We have then $\delta_x * \delta_y = \delta_{xy}$ and $\delta_x^* = \delta_{x^{-1}}$. It follows that, a $*$-representation of $C_c^\infty(G)$ is nothing else than a *unitary representation* of G, i.e. a morphism $\pi : G \to \mathcal{U}(H)$ where $\mathcal{U}(H)$ is the group of unitary elements in $\mathcal{L}(H)$.

If G is a Lie (or just locally compact) group, the same is true: $*$-representations of $C_c^\infty(G)$ are unitary representations $\pi : G \to \mathcal{U}(H)$ of G which are *strongly continuous*, i.e. for every $\xi \in H$, the map $x \mapsto \pi(x)(\xi)$ is continuous from G to H.

The case of a space. Let us next examine the case when G is just a manifold M.

In that case, $C_c^\infty(G)$ is a commutative algebra—and therefore $C^*(G)$ also. It is not difficult to see that its spectrum is M and therefore $C^*(G) = C_0(M)$.

Any Radon measure μ on M gives a representation of $C_0(M)$ by multiplication on the Hilbert space $L^2(M, \mu)$ of square integrable functions (modulo null ones).

Actually, every representation of $C_0(M)$ is given, up to multiplicity in that way. More precisely, every representation is *equivalent* to a representation given by

- a Radon measure μ on M and
- a μ-measurable (multiplicity) function $\varphi : M \to \mathbb{N} \cup \{+\infty\}$ (with $\varphi(x) \geq 1$ almost everywhere).

The Hilbert space identifies with a subspace $\mathcal{H}(\mu, \varphi)$ of the space $L^2(M, \mu) \otimes \ell^2(\mathbb{N})$ of square integrable functions from M to $\ell^2(\mathbb{N})$, up to functions vanishing almost everywhere:

For $k \in \mathbb{N} \cup \{+\infty\}$, let H_k be the Hilbert subspace of $\ell^2(\mathbb{N})$ (of dimension k) consisting of sequences (x_n) of complex numbers such that $x_n = 0$ for $n \geq k$ (and $\sum |x_n|^2 < +\infty$—this condition is always satisfied if $k \neq \infty$).

The space $\mathcal{H}(\mu, \varphi)$ consists of square integrable functions from M to $\ell^2(\mathbb{N})$ such that $\xi(x) \in H_{\varphi(x)}$, up to functions vanishing almost everywhere.

We write $\mathcal{H}(\mu, \varphi) = \int^\oplus H_{\varphi(x)}\, d\mu(x)$.

The general case. Let $G \rightrightarrows M$ be a Lie groupoid. A representation of G is given by

- a Radon measure on M and
- a μ-measurable (multiplicity) function $\varphi : M \to \mathbb{N} \cup \{+\infty\}$ (with $\varphi(x) \geq 1$ almost everywhere);
- a measurable map $x \mapsto U(x)$ from G to the set of unitaries on $\mathbb{C}^{\varphi(s(x))}$ (if $\varphi(s(x)) < +\infty$; on $\ell^2(\mathbb{N})$ if $\varphi(s(x)) = +\infty$).

They satisfy the following facts:

1. The measure μ is *quasi-invariant* : fixing a smooth Haar system ν on G, the Radon measures μ_r and μ_s on G given, for $f \in C_c(G)$, by $\mu_r(f) = \int_M \Big(\int_{G^u} f(x) d\nu^u(x) \Big) d\mu(x)$ and $\mu_s(f) = \int_M \Big(\int_{G^u} f(x^{-1}) d\nu_u(x) \Big) d\mu(x)$ are equivalent (i.e. have the same null sets). There exists then a *Radon-Nykodym derivative*: a measurable function $\Delta : G \to \mathbb{R}_+^*$ such that $\mu_r = \Delta \mu_s$— i.e. $\mu_r(f) = \mu_s(\Delta f) = \int_M \Big(\int_{G^u} f(x^{-1}) \Delta(x^{-1})\, d\nu_u(x) \Big) d\mu(x)$ for every f.
   The map Δ is easily seen to be a "cocycle", i.e. a morphism of groupoids: we have $\Delta(xy) = \Delta(x)\Delta(y)$ for almost all $(x, y) \in G^{(2)}$ (recall $G^{(2)}$ is the set of composable arrows).
2. The multiplicity φ is constant along the G-orbits: $\varphi \circ r = \varphi \circ s$ (almost everywhere).
3. The map U is a representation: we have $U(xy) = U(x)U(y)$ for almost all $(x, y) \in G^{(2)}$.

Representations of $C_c^\infty(G)$. To a representation (μ, φ, U) of G corresponds a representation π of $C_c^\infty(G)$ on $\mathcal{H}(\mu, \varphi)$ given by

$$\pi(f)(g)(u) = \int_{G^u} \Delta(x)^{-1/2} f(x)\, U(x) \big(g(s(x))\big)\, d\nu(x)$$

for $f \in C_c^\infty(G)$, and $g \in \mathcal{H}(\mu, \varphi)$.

C^*-Algebra of a Lie Groupoid

The C^*-algebra $C^*(G)$ of a groupoid G is a completion of the $*$-algebra $C_c^\infty(G)$ with respect to a norm.

We just put $\|f\| = \sup \|\pi(f)\|$, the supremum is taken over all the representations of the groupoid G described above.

Remark. Actually, although I don't really wish to enter the details, let me mention that there is another natural C^*-completion of $C_c^\infty(G)$, called the reduced C^*-algebra of G and denoted by $C_r^*(G)$ or $C_{red}^*(G)$, where we only take the supremum of $\|\lambda_u\|$ for $u \in M$, where λ_u is the natural 'regular' representation—by convolution—on $L^2(G_u)$ (where $G_u = \{x \in G;\ s(x) = u\}$).

Examples

1. If G is just a space $M \rightrightarrows M$, then of course convolution is just multiplication of functions, and $C^*(G)$ is the C^*-algebra of continuous functions on M which vanish at infinity if M is not compact.
2. If G is a commutative group, then $C^*(G)$ is immediately seen to be commutative and therefore it is the algebra of continuous functions on its spectrum (vanishing at infinity if $C^*(G)$ is not unital, i.e. when the group G is not discrete). The spectrum of $C^*(G)$ is the set of its characters, i.e. its representations of dimension 1. We have described the representations of $C^*(G)$, and it follows that a character of $C^*(G)$ is nothing else than a continuous morphism from G to the group $\mathbb{U}(1)$ of complex numbers of modulus 1. This set has a natural structure of locally compact group called the *Pontryagin dual* of G, and denoted by $\widehat{G}$: in other words, we have $C^*(G) = C_0(\widehat{G})$.
   If G is a finite dimensional vector space, the Pontryagin dual $\widehat{G}$ of G identifies to the dual vector space G^* of G through the morphism $\iota : G^* \to \widehat{G}$ defined by setting $\iota(\ell)(x) = e^{i\ell(x)}$ for $\ell \in G^*$ and $x \in G$. The isomorphism $C^*(G) = C_0(G^*)$ is in fact given by Fourier transform: the image of $f \in L^1(G) \subset C^*(G)$ in $C^0(G)$ is $\widehat{f}$.
   More generally, if G is a vector bundle $G \to M$, we find $C^*(G) = C_0(G^*)$, where G^* denotes the (total space of the) dual vector bundle.

   The above groupoids give rise to commutative C^*-algebras. In fact, it is not difficult to see that, the C^*-algebra $C^*(G)$ of a groupoid $G \rightrightarrows M$ is commutative if and only if $r = s$ and, for every u in M, the group $G_u = \{x \in G;\ s(x) = u(= r(x))\}$ is commutative.
3. $(*)$ At the opposite, consider the pair groupoid $M \times M \overset{r,s}{\rightrightarrows} M$; kernels $f \in C_c^\infty(M \times M)$, since they are in $L^2(M \times M)$ give rise to Hilbert-Schmidt operators—and therefore elements of $\mathcal{K}(L^2(M))$. It follows easily that $C^*(M \times M) = \mathcal{K}(L^2(M))$.
4. A more general situation is the 'family of pair groupoids': given smooth manifolds M and B and a smooth surjective submersion $\pi : M \to B$, we can consider the groupoid $M \times_B M = \{(x, y) \in M \times M;\ \pi(x) = \pi(y)\}$, which is a smooth submanifold and a subgroupoid of the groupoid $M \times M$. One can think of it as the family indexed by $b \in B$ of the pair groupoids $\pi^{-1}(\{b\}) \times \pi^{-1}(\{b\})$ of the *fiber* $\pi^{-1}(\{b\})$. Its C^*-algebra is the algebra of continuous families of compact operators on L^2 of the fiber.
   $(*)$ In fact, the field of the Hilbert spaces $(L^2(\pi^{-1}(\{b\})))_{b\in B}$ is trivial (if the fibers are of nonzero dimension): it is isomorphic to the constant field $\ell^2(\mathbb{N})$ and therefore $C^*(M \times_\pi M)$ is the algebra $C(B) \otimes \mathcal{K} = C(B; \mathcal{K})$ of continuous functions from B to $\mathcal{K}(\ell^2(\mathbb{N}))$.

2.5.3 Exact Sequences and Analytic Index

Exact Sequence Associated with an Open Saturated Subset

Let $G \rightrightarrows M$ be a Lie groupoid. A subset $A \subset M$ is said to be *saturated* or *G-invariant* if for every $x \in G$ such that $s(x) \in A$, we have $r(x) \in A$.

Let now $U \subset M$ an open and saturated subset. Denote by F the complement $F = M \setminus U$ of U in M.

The set $G_U = \{x \in G;\ s(x) \in U\}$ is an open subset and sub-groupoid—and therefore a Lie groupoid. The functions in $C_c^\infty(G_U)$ can be extended by 0 and give elements in $C_c^\infty(G)$. In this way, $C_c^\infty(G_U)$ sits as a two-sided ideal in the in $C_c^\infty(G)$. It follows that $C^*(G_U)$ sits as a two-sided ideal in the in $C^*(G)$.

The quotient C^*-algebra $C^*(G)/C^*(G_U)$ identifies with the C^*-algebra of the groupoid $G_F = \{x \in G;\ s(x) \in F\}$.[4]

We thus obtain an exact sequence

$$0 \to C^*(G_U) \to C^*(G) \to C^*(G_F) \to 0.$$

The Tangent Groupoid and Analytic Index

Let us first note that, for every C^*-algebra A, the C^*-algebra

$$B = C_0((0, 1]; A) = \{f : [0, 1] \to A;\ \text{ continuous, } f(0) = 0\}$$

is *contractible*, i.e. the identity map of B is homotopic to the 0-map. Indeed, for $t \in [0, 1]$, let $\alpha_t : B \to B$ be the endomorphism defined by $(\alpha_t(f))(s) = f(st) \in A$ (for $f : [0, 1] \to A$ with $f(0) = 0$ and $s \in [0, 1]$). Note that α_1 is the identity of B and α_0 is the 0 map. It follows that $K_0(B) = K_1(B) = \{0\}$.

Given a deformation groupoid of the type $\mathcal{G} = G_0 \times \{0\} \times G_1 \times (0, 1]$, we obtain an exact sequence

$$0 \to C_0((0, 1]; C^*(G_1)) \to C^*(\mathcal{G}) \to C^*(G_0) \to 0.$$

As $C_0((0, 1]; C^*(G_1))$ is contractible, using the six term exact sequence of K theory it follows that the morphism $ev_0 : C^*(\mathcal{G}) \to C^*(G_0)$ of evaluation at 0 induces an isomorphism $(ev_0)_*$ in K theory.

We thus find a morphism $\mathrm{ind}_a = \partial_G = (ev_1)_* \circ (ev_0)_*^{-1} : K_i(C^*(G_0)) \to K_i(C^*(G_1))$.

[4] If F is a submanifold, G_F is a Lie groupoid. One can in fact define the C^*-algebra of G_F for every (locally) closed saturated subset F of M.

Taking now Connes' tangent groupoid $G_T = M \times M \times (0,1] \sqcup TM \times \{0\}$, we obtain an exact sequence

$$0 \to C_0((0,1]; \mathcal{K}) \to C^*(G_T) \to C^*(TM) = C_0(T^*M) \to 0,$$

the corresponding morphism $\partial_{G_T} : K_0(C^*(TM)) = C_0(T^*M) \to K_0(\mathcal{K}) = \mathbb{Z}$ is the *analytic index map.*

More generally, given a Lie groupoid G, and using the deformation (to the normal cone) groupoid $G_{ad} = G \times (0,1] \sqcup \mathfrak{A}G \times \{0\}$ associated with the inclusion $G^{(0)} \to G$, we obtain the *analytic index map*

$$\mathrm{ind}_a^G = \partial_{G_{ad}} : K_0(C^*(\mathfrak{A}G)) = C_0(\mathfrak{A}^*G) \to K_0(C^*(G))$$

for the groupoid G.

Remark 5 More generally, if $\Gamma \subset G$ is a Lie subgroupoid of G, the deformation (to the norma cone) groupoid $N_\Gamma^G \sqcup G \times (0,1]$ gives rise to a morphism $\partial_\Gamma^G : K_*(C^*(N_\Gamma^G)) \to K_*(C^*(G))$.

Let us compute the groupoid N_Γ^G in the particular case when Γ is just a submanifold $V \subset M = G^{(0)}$.

The Inverse of the Thom Isomorphism

Let $p : E \to X$ be a vector bundle over a locally compact space. We can form a "Thom" groupoid $\mathcal{G}_E \rightrightarrows E \times [0,1]$ by setting $\mathcal{G}_E = E \times_p E \times [0,1]$ putting $s(\xi, \eta, t) = (\eta, t), r(\xi, \eta, t) = (\eta + t\xi, t)$ and $(\xi_1, \eta_1, t)(\xi_2, \eta_2, t) = (\xi_1 + \xi_2, \eta_2, t)$ (when $\eta_1 = \eta_2 + t\xi_1$). In this way, we obtain a locally compact groupoid, which is a Lie groupoid if X is a manifold and E is a smooth vector bundle. Its restriction to the open saturated subset $X \times (0,1]$ identifies to the family of pair groupoids $E \times_p E \times (0,1]$ under the map $(r,s) : (\xi, \eta, t) \mapsto (t\xi + \eta, \eta, t)$.

In other words $\mathcal{G}_E$ is a deformation groupoid and it defines a morphism $\partial_{\mathcal{G}_E} : K_*(C^*(\mathcal{G}_E^0)) \to K_*(C^*(\mathcal{G}_E^1))$ where $C^*(\mathcal{G}_E^i)$ is the restriction of $\mathcal{G}_E$ to the closed saturated subset $E \times \{i\} \subset E \times [0,1]$ (for $i = 0$ or 1).

The restriction to $\mathcal{G}_E^0 \rightrightarrows E \times \{0\}$ is the vector bundle $p^*(E) \rightrightarrows E$. Its C^*-algebra is the commutative C^*-algebra $C_0(E \oplus E^*)$. Note that the bundle $E \oplus E^*$, using any hermitian metric on E, identifies with the bundle $E + iE$, i.e. the complexification $E_{\mathbb{C}}$ of E.

The restriction to $\mathcal{G}_E^1 \rightrightarrows E \times \{1\}$ is the family of pair groupoids $E \times_X E$. Its C^*-algebra is Morita equivalent to $C_0(X)$.

Under these identifications, the map

$$\partial_{\mathcal{G}_E} : K_*(C_0(E_{\mathbb{C}})) = K_*(C^*(\mathcal{G}_E^0 \rightrightarrows E \times \{0\})) \to K_*(C^*(\mathcal{G}_E^1 \rightrightarrows E \times \{1\})) = K_*(C_0(X))$$

is the inverse of the Thom isomorphism for the complexified bundle $E_{\mathbb{C}}$.

2.6 Differential and Pseudodifferential Operators

The true and original definition of the analytic index uses the pseudodifferential calculus. We outline it here and explain how it relates with our definition above although it is not really needed for the index theorem of foliations in Sect. 3.3. Therefore, the present section is independent of the rest of this chapter. It is only here that we will use the analysis of distributions.

Here, we will very briefly introduce some material around the pseudodifferential calculus. The interested reader will find more precise statements and proofs in the quite abundant literature on the subject. See [1, 15, 25, 27].

2.6.1 A Few Words on Distributions

Let M be a smooth manifold. A *distribution* on M is an element of the topological dual of $C_c^\infty(M)$, i.e. a linear form $\chi : C_c^\infty(M) \to \mathbb{C}$ which is continuous with the natural topology.

If M is compact, this means that there exists $k \in \mathbb{N}$ such that χ extends to a continuous linear form on the Banach space $C^k(M)$ of functions of class C^k.

If M is not compact, this means that for every compact set $K \subset M$, there exists $k \in \mathbb{N}$ such that χ extends to a continuous linear form on the Banach space $C^k(M, K)$ of functions of class C^k that vanish out of M.

The *support* of a distribution χ is the smallest closed subset S of M such that $\chi(f) = 0$ if the restriction of f to S is 0.

Fix a volume form dx on M. A smooth function f on M defines a distribution χ_f by setting $\chi_f(g) = \int f(x)\, g(x)\, dx$.

One sees (by "smearing" χ) that every distribution χ is a limit of smooth functions as above: there exists a (generalised) sequence (f_n) of smooth functions on M such that, for every $g \in C_c^\infty(M)$, $\chi(f) = \lim \int f_n(x)\, g(x)\, d\mu(x)$.

With this remark in hand, one thinks of distributions as *generalised functions* on M, and writes $\langle \chi \,|\, f \rangle = \int_M \chi(x)\, f(x)\, dx$.

The *singular support* of a distribution χ is the smallest closed subset S of M such that there exists a smooth function f on $M \setminus S$ satisfying $\langle \chi | g \rangle = \int f(x)\, g(x)\, dx$ for every $g \in C_c^\infty(M \setminus K)$—i.e. $g \in C_c^\infty(M)$ vanishing in a neighbourhood of S.

2.6.2 Pseudodifferential Calculus on a Manifold

A *differential operator* on $\mathbb{R}^n$ is a map $D : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ which is a finite sum $\sum_\alpha f_\alpha \partial^\alpha$ where $\alpha \in \mathbb{N}^n$ is a multi index $\alpha = (\alpha_1, \ldots, \alpha_n)$, f_α is a smooth function on $\mathbb{R}^n$ and ∂^α is defined by partial derivatives: $\partial^\alpha(g) = \dfrac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{(\partial x_n)^{\alpha_n}} g$.

The Fourier transform of $\dfrac{\partial g}{\partial x_j}$ is the function $(\xi_1, \ldots, \xi_n) \mapsto (i\xi_j)\hat{g}(\xi_1, \ldots, \xi_n)$ (for g smooth with compact support—or a Schwartz function). One may write

$$(Dg)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x|\xi\rangle} P(x, \xi)\hat{g}(\xi)\, d^n\xi$$

where $P(x, \xi) = \sum_\alpha f_\alpha(x)(i\xi_1)^{\alpha_1} \ldots (i\xi_n)^{\alpha_n}$. The function P, which is smooth in x and polynomial in ξ, is called the *symbol* of D.

Classical Symbols

This formula makes sense for many more symbols—i.e. functions $P(x, \xi)$. We will use here the so called "classical" symbols.

Definition 14 A classical symbol on $\mathbb{R}^n$ is a function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$.

- It has an *order* $m \in \mathbb{Z}$.
- We assume that we have an asymptotic expansion (for ξ large) $a(x, \xi) \sim \sum_{k=0}^{+\infty} a_{m-k}(x, \xi)$,[5] where a_ℓ is homogeneous of order ℓ in ξ, i.e. $a_\ell(x, t\xi) = t^\ell a(x, \xi)$ for $t \in \mathbb{R}_+^*$.

We denote by $\mathcal{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ the space of classical symbols of order n on $\mathbb{R}^n$.

If $a \in \mathcal{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ is a classical symbol on $\mathbb{R}^n$, and f is a smooth function with compact support on $\mathbb{R}^n$, we put

$$(P_a(f))(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x|\xi\rangle} a(x, \xi)\hat{f}(\xi)\, d^n\xi\,.$$

---

[5] This expansion means that, for every $k \in \mathbb{N}$, the remainder $R_k(x, \xi) = a(x, \xi) - \sum_{j=0}^{k-1} a_{m-j}(x, \xi)$, satisfies that for every compact subset $K \subset \mathbb{R}^n$, for every $\alpha, \beta \in \mathbb{N}^n$, we have

$$\sup_{x \in K;\ \|\xi\| \geq 1} \|\xi\|^{m-k-|\beta|} \left| \partial_x^\alpha \partial_\xi^\beta R_k(x, \xi) \right| < +\infty.$$

One sees easily (using derivatives of functions defined by integrals) that the function $P_a(f)$ is a smooth function. Writing explicitly the Fourier transform $\hat{f}(\xi)$, we find

$$(P_a(f))(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} e^{i\langle x-y|\xi\rangle} a(x,\xi) f(y)\, d^n y \Big)\, d^n \xi\,.$$

Definition 15 The top homogeneous part a_m of the symbol a is called the *principal symbol* of the operator P_a.

Conormal Distributions

We can also write $P_a(f)(x) = \lim_{R\to\infty} \int k_{a,R}(x,y) f(y)\, d^n y$ where

$$k_{a,R}(x,y) = \frac{1}{(2\pi)^n} \int_{\xi\in\mathbb{R}^n,\ \|\xi\|\le R} e^{i\langle x-y|\xi\rangle} a(x,\xi)\, d^n\xi\,.$$

Put $P_a^R(f)(x) = \int k_{a,R}(x,y) f(y)\, d^n y$; the function $P_a^R(f)$ converges to $P_a(f)$ in the natural topology of C^∞. Now, $k_{a,R}$ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$. It converges to a *distribution* on $\mathbb{R}^n \times \mathbb{R}^n$. Denoting by k_a this distribution we have $P_a(f)(x) = \int k_a(x,y) f(y)\, d^n y$.

We write formally $k_a(x,y) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{\xi\in\mathbb{R}^n} e^{i\langle x-y|\xi\rangle} a(x,\xi)\, d^n\xi\,.$

The distribution k_a written in this way is smooth near any point (x,y) with $x \neq y$.

The kind of distributions on $\mathbb{R}^n \times \mathbb{R}^n$ given as above are called distributions on $\mathbb{R}^n \times \mathbb{R}^n$ *conormal* to the diagonal $\Delta = \{(x,x);\ x \in \mathbb{R}^n\}$.

Let X be a smooth manifold and $Y \subset X$ a submanifold. Let $N = N_Y^X$ be the normal bundle of the inclusion $Y \subset X$ and N^* its dual bundle. A symbol a on N^*, i.e. a function $a : N^* \to \mathbb{C}$ which has an asymptotic extension $a(x,\xi) \sim \sum_{k=0}^{+\infty} a_{m-k}(x,\xi)$, defines by Fourier transform a distribution $\hat{a}$ on N which is smooth outside the 0 section $Y \subset N$.

A tubular neighborhood construction (see Sect. 2.1.3), gives a diffeomorphism $\varphi : U \to V$ of a neighborhood U of Y in X with a neighborhood V of Y in N. We then may construct a distribution on X which is smooth outside Y and coincides with $\hat{a} \circ \varphi$ near Y. Such a distribution is called a *conormal distribution*; the space of those distributions is denoted by $\mathcal{I}(X,Y)$. We denote also by $\mathcal{I}_c(X,Y)$ the subspace of $\mathcal{I}(X,Y)$ of conormal distributions with compact support.

It turns out that two different tubular neighborhood constructions give similar results:

- The space $\mathcal{I}_c(X, Y)$ does not depend on the choice of local diffeomorphisms $\varphi : N_Y^X \to X$ (which are tubular neighborhood constructions); given φ_1 and φ_2, for every symbol a_1 on $(N_Y^X)^*$, there exists a symbol a_2 on $(N_Y^X)^*$ such that, $\widehat{a_1} \circ \varphi_1^{-1}$ coincides with $\widehat{a_2} \circ \varphi_2^{-1}$ in a neighborhood of Y.
- The principal symbol is well defined: with the notation above, a_1 and a_2 have the same top part. This is called the *principal symbol* of the distribution $\widehat{a_1} \circ \varphi_1^{-1} \simeq \widehat{a_2} \circ \varphi_2^{-1}$.

Pseudodifferential Operators on a Manifold

Pseudodifferential operators on $\mathbb{R}^n$ are conormal distributions, i.e. elements of $\mathcal{I}(\mathbb{R}^n \times \mathbb{R}^n; \Delta_{\mathbb{R}^n})$ where $\Delta_{\mathbb{R}^n} = \{(x, x)\,,\ x \in \mathbb{R}^n\}$.

It follows that for every smooth (compact) manifold M, a conormal distribution $\chi \in \mathcal{I}(M \times M; \Delta_M)$ defines an operator $P : C^\infty(M) \to C^\infty(M)$ by a formula $(Pf)(x) = \displaystyle\int_M \chi(x, y)\, f(y)\, dy$.

(∗) **Remark.** A priori, starting with a general distribution χ on $M \times M$, the formula $(\chi * f)(x) = \displaystyle\int_M \chi(x, y) f(y)\, dy$ gives rise to a distribution on M given by $\langle \chi * f | g \rangle = \displaystyle\int_M (\chi * f)(x) g(x)\, dx = \langle \chi | f \otimes g \rangle$, where $f \otimes g(x, y) = f(x)g(y)$.
The claim is that if χ is a conormal distribution, then for every smooth function f on M, the distribution $\chi * f$ is a smooth function.

Definition 16 A pseudodifferential operator on M is an operator $f \mapsto \chi * f$ associated with a conormal distribution $\chi \in \mathcal{I}(M \times M; \Delta_M)$.

The principal symbol $\sigma_m(P)$ of a pseudodifferential operator P of order m is the principal symbol of the associated conormal distribution χ.

Moreover,

- The composition PQ of pseudodifferential operators P and Q is a pseudodifferential operator and $\sigma(PQ) = \sigma(P)\sigma(Q)$.
- The adjoint of a pseudodifferential operator P is a pseudodifferential operator of the same order.
- If the order of a pseudodifferential operator P is ≤ 0, then P extends to a continuous operator on the Hilbert space $L^2(M)$ of square integrable functions on M.
- If the order of a pseudodifferential operator P is < 0, then P defines a compact operator on $L^2(M)$.

We obtain an exact sequence of C^*-algebras:

$$0 \to \mathcal{K}(L^2(M)) \to \Psi^*(M) \xrightarrow{\sigma} C(S^*M) \to 0, \qquad (PDO_M)$$

where $\Psi^*(M)$ is the closure in $\mathcal{L}(L^2(M))$ of the algebra of pseudodifferential operators on M, S^*M is the space of half lines in T^*M which identifies with the sphere bundle $\{(x,\xi) \in T^*M;\ \|\xi\| = 1\}$ if we choose a euclidean metric on T^*_x for every $x \in M$ and σ is the principal symbol map σ_0. Note that a function on S^*M is a fonction on $T^*M \setminus M$ which is homogeneous of order 0.

Some Classical Elliptic Differential Operators

Most important examples of elliptic operators connected with index theory are not *scalar* i.e. operators from $L^2(M)$ to $L^2(M)$, but rather operators from a vector bundle E^+ to a vector bundle E^-, i.e. maps from the space $L^2(M; E^+)$ of L^2 sections of E^+ to the space $L^2(M; E^+)$ of L^2 sections of E^-.

Moreover they are usually differential of order one—and therefore unbounded.

We briefly discuss here these two issues and give a few classical examples of elliptic operators.

Matrices and vector bundles. We may of course define a pseudodifferential matrix P (of order m) to be a matrix (P_{ij}) of pseudodifferential operators. Its principal symbol is the map $\sigma_P : T^*M \setminus M \to M_n(\mathbb{C})$, where $(\sigma_P)_{i,j} = \sigma_{P_{i,j}}$.

Now, given a complex vector bundle E over M, there exists a "complementary" complex vector bundle F such that the bundle $E \oplus F$ is a trivial bundle isomorphic to $M \times \mathbb{C}^n \to M$. Choosing any hermitian metric on this bundle, we write $E = p\mathbb{C}^n$ where p is a smooth map $x \mapsto p_x$ from M to $M_n(\mathbb{C})$ such that for every $x \in M$, p_x is a self-adjoint idempotent in $M_n(\mathbb{C})$.

Take two vector bundles $E^\pm$ and write them as $p^\pm\mathbb{C}^n$. A pseudodifferential operator of order m from E^+ to E^- is a pseudodifferential matrix such that $P = p^- P p^+$. It maps sections of E^+ to sections of E^- (and its adjoint, sections of E^- to sections of E^+).

Its principal symbol σ_P assigns to (x,ξ) (where $x \in M$ and $\xi \in T^*_x M$ non zero) a linear map $\sigma_P(x,\xi) : E^+_x \to E^-_x$. The operator P is elliptic if for every $x \in M$ and $\xi \in T^*_x M$, $\xi \neq 0$, the map $\sigma_P(x,\xi)$ is an isomorphism.

Let P be an elliptic pseudodifferential operator of order 0 from E^+ to E^-. By ellipticity of P, we find a pseudodifferential operator Q of order 0 from E^- to E^+ whose principal symbol σ_Q is defined by $\sigma_P(x,\xi) = \sigma_Q(x,\xi)^{-1}$. Then $\mathrm{id} - QP$ and $\mathrm{id} - PQ$ are pseudodifferential operators of strictly negative order from E^+ to E^+ and from E^- to E^-. It follows that P is a Fredholm operator and has therefore a Fredholm index.

The symbol of P defines an element of K-theory with compact supports of T^*M (cf. Remark 2.3.2).

Positive order elliptic operators. Let P be a pseudodifferential operator of order $m > 0$. Given a Riemannian metric on M, we may define a symbol of order 0 by setting $a(x,\xi) = \sigma_P(x, \frac{1}{\|\xi\|}\xi)$. The symbols σ_P and a are homotopic and thus define the same class in ${}^0(T^*M)$. Also, a pseudodifferential operator operator with symbol a has the same index as P.

It follows that the computations of indices for 0-order operators extend immediately to elliptic operators of any order.

Examples. The operator d of differentiation of functions is a differential operator. Its principal symbol is just given by $\sigma_d(x,\xi)(\lambda) = i\lambda\xi$. The operator d is not elliptic (unless $\dim M = 1$), but it is the starting point of many important elliptic differential operators.

Example: Euler characteristic One may extend the operator d to the bundle of exterior powers of T^*M. We call k-forms on M sections of the kth exterior power $\Lambda^k T^*M$ of T^*M ($0 \leq k \leq \dim M$). One then extends d to smooth k-forms in a way that $d(d(\alpha)) = 0$ and $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$ for every k, ℓ, every k-form α and every ℓ-form β.

The operator d thus extended is called the *exterior differentiation*. It is a differential operator of order 1 and its principal symbol is given by $\sigma_d(x,\alpha)(\alpha) = i\xi \wedge \alpha$.

The operator $d + d^*$ is elliptic—its square $(d + d^*)^2 = \Delta$ is the (positive) Laplacian on forms whose principal symbol is $\sigma_\Delta(x,\xi)(\alpha) = \|\xi\|^2\alpha$. The principal symbol of $d + d^*$, given by $\sigma_{d+d^*}(x,\xi) = \sigma_d(x,\xi) + \sigma_{d^*}(x,\xi) = (i\xi\wedge) + (i\xi\wedge)^*$ is called the *Clifford multiplication* by ξ.

The kernel of $d + d^*$ is the kernel of $\Delta = (d + d^*)^2$: the space of harmonic forms and identifies with the de Rham cohomology of M. Its dimension is $\sum_{k=0}^{\dim M} \beta_k(M)$ where β_k is the kth Betti number of M, i.e. the dimension of the kth homology space of M.

Of course the operator $d + d^*$ is self-adjoint and thus its index is 0.

On the other hand it can be seen as an elliptic operator from even forms to odd forms and its index is then $\sum_{k=0}^{\dim M} (-1)^k \beta_k(M)$, called the Euler characteristic of M.

Example: the signature operator When the riemannian manifold M is even dimensional—write $\dim M = 2\ell$, and *oriented*, one constructs another natural way of splitting $\Lambda^* T^*M = \bigoplus_{k=0}^{\dim M} \Lambda^k T^*M$ as a direct sum $E^+ \oplus E^-$ of vector bundles in such a way that $d + d^*$ maps section of $E^\pm$ to sections of $E^\mp$.

Indeed, the bundle $\Lambda^{2\ell} T^*M$ is trivial thanks to the orientation and the Riemannian structure on M. Using this trivialisation, we define the *Hodge* operator $\star$ (mapping $\Lambda^k T^*M$-forms to $\Lambda^{2\ell-k}$) by setting $\langle\alpha|\beta\rangle = \overline{\alpha} \wedge (\star\beta)$. The Hodge operator $\star$ is isometric and $\star^2(\alpha) = (-1)^k\alpha$ for $\alpha \in \Lambda^k T^*M$.

Using the formulas $\mathrm{ind}_M d\omega = 0$ (for any $2\ell - 1$ form ω) and $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$ (for a k form α), we find, for a k form α and a $k+1$-form β,

$$\begin{aligned}\langle\alpha|d^*\beta\rangle = \langle d\alpha|\beta\rangle &= \int_{x\in M} \overline{d\alpha_x} \wedge (\star\,\beta_x)\\ &= (-1)^{k+1}\int_{x\in M} \overline{\alpha_x} \wedge (d\star\beta)_x\\ &= (-1)^{k+1}\langle\alpha|\star^{-1}(d(\star\beta))\rangle\\ &= -\langle\alpha|\star\, d\star\,\beta\rangle\end{aligned}$$

We find $d^* = -\star\, d\,\star$. One slightly modifies $\star$ by setting $\tau(\alpha) = i^{k(k-1)+\ell}\star\alpha$ for $\alpha \in \Lambda^k T^*M$. We still have $d^* = -\tau d\,\tau$ but now $\tau^2 = \mathrm{id}$.

The bundles $E^{\pm}$ are just the eigenspaces for the eigenvalues ± 1 of τ.

Since $d^* = -\tau d\tau$ and $\tau^2 = 1$, it follows that $d = -\tau d^*\tau$ and $(d + d^*) = -\tau(d + d^*)\tau$; therefore $d + d^*$ defines en elliptic differential operator from sections of E^+ to sections of E^-.

Its index is 0 when ℓ is odd; if ℓ is even, i.e. if the dimension of M is $4k$, this index identifies easily with the signature of the symmetric bilinear form $(\alpha, \beta) \mapsto \int \alpha \wedge \beta$ on the space $H^\ell(M, \mathbb{R})$.

Example: the case of a quasi complex manifold A manifold M is said to be *quasi complex* if $\dim M$ is even, say 2ℓ and the tangent bundle TM is endowed with a structure of a complex vector bundle.

In that case, T^*M carries also a complex structure, i.e. it is isomorphic to a complex vector bundle E of dimension ℓ. We can then construct

- the exterior algebra $\Lambda^* E$ of this complex bundle (of complex dimension 2^ℓ);
- a differential operator D acting on the sections of the bundle $\Lambda^* E$ whose principal symbol is given by $\sigma_D(x, \xi) = (\xi\wedge) + (\xi\wedge)^*$—where $(\xi\wedge)(\alpha) = \xi \wedge \alpha$ (for $\alpha \in \Lambda^* E_x$).

The operator D defines an elliptic operator from even to odd forms.

Remark 6 Note that given any complex vector bundle E over a compact space X, we can consider the element $(x, \xi) \mapsto c(x, \xi)$ acting on $\Lambda^* E$ by setting as above $c(x, \xi) = (\xi\wedge) + (\xi\wedge)^*$. It defines the element $(\Lambda^{even} E, \Lambda^{odd} E, c)$ in the K-theory of E with compact supports (see Remark 2.3.2). This actually describes the Thom isomorphism: to the class of a complex vector bundle F over M, we associate the triple $(\Lambda^{even} E \otimes F\ \Lambda^{odd} E \otimes F, c_F)$ where $c_F(x, \xi) = c(x, \xi) \otimes \mathrm{id}_{F_x}$.

2.6.3 Pseudodifferential Calculus on Groupoids

In the same way, one defines a pseudo-differential operators on groupoids to be given by conormal distributions, i.e. elements of $I_c(G, G^{(0)})$.

If χ is a distribution on G and $f \in C_c^\infty(G)$, we may define $\chi * f$ to be *a priori* a distribution on G by setting formally $\chi * f(x) = \int_{yz=x} \chi(y) f(z)\, d\nu$ which reads, for $g \in C_c^\infty(G)$,

$$\langle \chi * f | g \rangle = \int \chi * f(x)\, g(x)\, dx = \langle \chi | g * \tilde{f} \rangle,$$

where $\tilde{f}(y) = f(y^{-1})$.

Similarly the distribution $f * \chi$ is defined by setting $\langle f * \chi | g \rangle := \langle \chi | \tilde{f} * g \rangle$.

Now, one proves the following facts.

- If $\chi \in I_c(G, G^{(0)})$, then for every $f \in C_c^\infty(G)$, the convolutions $\chi * f$ and $f * \chi$ are smooth functions on G.
- For every $\chi \in I_c(G, G^{(0)})$, every $f \in C_c^\infty(G)$, and every $u \in G^{(0)}$, the restriction of the smooth function $\chi * f$ to G_u only depends on the restriction of f to G_u. The map $f \mapsto \chi * f$ defines a family of pseudodifferential operators $(P_u)_{u \in G^{(0)}}$. This family is *invariant*: an $x \in G$, defines a diffeomorphism $y \mapsto yx$ from $G_{r(x)}$ to $G_{s(x)}$ and therefore an isomorphism $U_x : C_c^\infty(G_{s(x)}) \to C_c^\infty(G_{r(x)})$; we have $P_{r(x)} \circ U_x = U_x \circ P_{s(x)}$.
- If $\chi_1, \chi_2 \in I_c(G, G^{(0)})$ then one may define the distribution $\chi_1 * \chi_2$ by setting

$$\langle \chi_1 * \chi_2 | f \rangle = \langle \chi_1 | f * \widetilde{\chi_2} \rangle = \langle \chi_2 | \widetilde{\chi_1} * f \rangle$$

  (where $\tilde{\chi}$ is defined by $\langle \tilde{\chi} | f \rangle = \langle \chi | \tilde{f} \rangle$).
  The distribution $\chi_1 * \chi_2$ is conormal (i.e. $\chi_1 * \chi_2 \in I_c(G, G^{(0)})$) and its principal symbol is $\sigma(\chi_1)\sigma(\chi_2)$.
- The adjoint of $\widetilde{\chi}$ of χ is a conormal distribution of the same order.
- if the order of a χ is < 0, then χ defines an element in $C^*(G)$.
- If the order of χ is ≤ 0, then χ is bounded (i.e. $f \mapsto \chi * f$—and $f \mapsto f * \chi$ extend to continuous linear maps from $C^*(G)$ to $C^*(G)$).
- Finally, the completion of the algebra of ≤ 0 order operators (with respect to the norm of the map $f \mapsto \chi * f$ from $C^*(G)$ to $C^*(G)$) is a C^*-algebra $\Psi^*(G)$ and we obtain an exact sequence of C^*-algebras:

$$0 \to C^*(G) \to \Psi^*(G) \xrightarrow{\sigma} C(S^*\mathfrak{A}_G) \to 0, \qquad (PDO_G)$$

  where $S^*\mathfrak{A}_G$ is the sphere bundle of the dual bundle to the algebroid $\mathfrak{A}_G$ and σ is the principal symbol map.

The connecting map of this exact sequence is a map

$$\partial_G : K_*(C(S^*\mathfrak{A}_G)) = K^*(S^*\mathfrak{A}_G) \to K_{*+1}(C^*(G)).$$

The map from $K_1(C(S^*\mathfrak{A}_G)) \to K_0(C^*(G))$ associates to an order zero pseudodifferential operator P whose principal symbol is invertible the class of P in $K_0(C^*(G))$.

An operator whose principal symbol is invertible is said to be *elliptic*. Let us say some further words on what is this index of an elliptic operator in the cases of the original papers of Atiyah-Singer [3, 4].

- If G is $M \times M$, an elliptic operator is Fredholm and the analytic index is the *Fredholm index* $\dim(\ker P) - \dim(\ker P^*)$.
- If $G = M \times_B M$, this index is a formal differences of vector bundles on B. More precisely, if $\dim(\ker P_b)$ does not depend on $b \in B$, the families $(\ker P_b)_{b\in B}$ and $(\ker P_b^*)_{b\in B}$ are vector bundles over B. In general, one can modify P keeping the same principal symbol so that $\dim(\ker P_b)$ is constant (see [4]).

Allowing vector bundles on $G^{(0)}$ or using relative K-theory we can refine a little bit this map to obtain the definition of the analytic index map

$$\widetilde{\mathrm{ind}}_a^G : K^*(\mathfrak{A}_G^*) = K_*(C_0(\mathfrak{A}_G^*)) \to K_*(C^*(G)).$$

It is rather easy to show that the analytic index defined in this way coincides with the analytic index defined in Sect. 2.5.3 ([8, 22, 23]). One just notices that the analytic index defined via pseudodifferential operators satisfies the following two properties.

Proposition 5 *1. Let $G \rightrightarrows M$ be a Lie groupoid and let $V \subset M$ be a closed* saturated *submanifold of M. Denote by G_V the restriction of G to V. We have a commutative diagram*

$$\begin{array}{ccc} K^0(\mathfrak{A}_G^*) & \longrightarrow & K^0(\mathfrak{A}_{G_V}^*) \\ \downarrow{\scriptstyle \widetilde{\mathrm{ind}}_a^G} & & \downarrow{\scriptstyle \widetilde{\mathrm{ind}}_a^{G_V}} \\ K_0(C^*(G)) & \longrightarrow & K_0(C^*(G_V)) \end{array}$$

2. Consider a vector bundle $E \to M$ as a groupoid—a family of groups. Then $\mathfrak{A}_E = E$ and, via Fourier transform, $C^*(E) = C_0(E^*)$ and $K_0(C_0(E^*)) = K^0(E^*)$. Then $\widetilde{\mathrm{ind}}_E : K^0(E^*) \to K_0(C^*(E))$ is the identity.

Proof An elliptic pseudodifferential P operator on G restricts to an elliptic pseudodifferential operator on G_V—whose symbol is the restriction to $\mathfrak{A}_{G_V}^*$ of the symbol of P. The first statement follows.

The second statement comes from the fact that the pseudodifferential calculus on a vector space (seen as a Lie group) is up to Fourier transform the multiplication by the symbol—and therefore the same holds for a vector bundle seen as a family of Lie groups. □

Theorem 14 ([8, 22, 23]) *Let G be a Lie groupoid. The analytic index defined via the pseudodifferential calculus coincides with the analytic index defined* via *deformation groupoids.*

Proof Let $\mathcal{G} = \mathfrak{A}_G \times \{0\} \sqcup G \times (0, 1]$ be the deformation groupoid. We have $\mathfrak{A}_{\mathcal{G}} = \mathcal{A}_G \times [0, 1]$. By the first statement of Proposition 5, we have a commutative diagram where the vertical arrows are analytic indices defined via pseudodifferential calculus:

$$\begin{array}{ccccc} K^0(\mathfrak{A}_G^*) & \xleftarrow{e_0} & K^0(\mathfrak{A}_G^* \times [0,1]) & \xrightarrow{e_1} & K^0(\mathfrak{A}_{G_V}^*) \\ \downarrow{\scriptstyle \widetilde{\mathrm{ind}}_a^{\mathfrak{A}_G}} & & \downarrow{\scriptstyle \widetilde{\mathrm{ind}}_a^{\mathcal{G}}} & & \downarrow{\scriptstyle \widetilde{\mathrm{ind}}_a^{G}} \\ K_0(C^*(\mathfrak{A}_G)) & \xleftarrow{ev_0} & K_0(C^*(\mathcal{G})) & \xrightarrow{ev_1} & K_0(C^*(G)) \end{array}$$

Note that, by homotopy invariance of K-theory, the maps e_0 and e_1 are equal and bijections.

By the second statement of Proposition 5, the map $\widetilde{\mathrm{ind}}_a^{\mathfrak{A}_G}$ is the identity. It follows that

$$\begin{aligned} \widetilde{\mathrm{ind}}_a^{G} &= ev_1 \circ \widetilde{\mathrm{ind}}_a^{\mathcal{G}} \circ e_1^{-1} \\ &= ev_1 \circ ev_0^{-1} \circ \widetilde{\mathrm{ind}}_a^{\mathfrak{A}_G} \circ e_0 \circ e_1^{-1} \\ &= ev_1 \circ ev_0^{-1} \\ &= \mathrm{ind}_a^{G} \end{aligned}$$

□

3 Foliations

3.1 Foliations: Definitions and Examples

3.1.1 Foliation Charts

A *foliation* on a manifold M is an equivalence relation on M. The equivalence classes are called the *leaves*. The leaves are assumed to be path connected embedded submanifolds of M.

Every point x of M admits an open neighborhood Ω which admits a diffeomorphism $\varphi : \Omega \to U \times T$ where U is a connected open subset of $\mathbb{R}^p$ and T is an open subset of $\mathbb{R}^q$. For every $t \in T$ the set $\varphi^{-1}(U \times \{t\})$ is called a *plaque* and is contained in a leaf and open for the topology of the leaf.

Let us give a more formal definition.

Definition 17 Let M be a manifold of dimension $p + q$. Two charts $\varphi_1 : \Omega_1 \to U_1 \times T_1$ and $\varphi_2 : \Omega_2 \to U_2 \times T_2$ (where U_i is a connected open subset of $\mathbb{R}^p$ and T_i is an open subset of $\mathbb{R}^q$) are said to be *compatible* if the diffeomorphism $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\Omega_1 \cap \Omega_2) \to \varphi_2(\Omega_1 \cap \Omega_2)$ is of the form $(u, t) \mapsto (g(u, t), h(t))$.[6]

[6] If $\Omega_1 \cap \Omega_2 = \emptyset$, the charts are compatible.

A *foliation of dimension p and co-dimension q* is described by a family $(\varphi_i)_{i\in I}$ of pairwise compatible charts of the form $\varphi_i : \Omega_i \to U_i \times T_i$ such that $\bigcup_i \Omega_i = M$.

Remark 7 Two families $(\varphi_i)_{i\in I}$, and $(\psi_j)_{j\in J}$ define the same foliation if for every $i \in I$ and $j \in J$, the charts φ_i and ψ_j are compatible.

3.1.2 Examples of Foliations

- A product is of course foliated. If $M = U \times T$, we can take the leaves to be (the connected components of) $U \times \{t\}$.
  More generally a submersion $p : M \to T$ gives rise to a foliation with leaves (the connected components of) the fibers $p^{-1}(\{t\})$. In particular a fibration gives rise to a foliation.
- Locally free Lie group actions. Let G be a connected Lie group acting on a manifold M; we assume that, for every $x \in M$ the map $g \mapsto g \cdot x$ is an immersion from $G \to M$. Then the orbits of G form a foliation on M.

Let us also mention some general and natural constructions of foliations

1. ***Quotients:*** We may start with a "trivial foliation" on a (non compact) manifold W and then take a quotient by a (discrete) group Γ acting *smoothly*, *freely* and *properly* on W in a *foliation preserving* way. We then obtain a foliation on the quotient space W/Γ. Before we give examples, let us say a few words about the conditions on the action:

   Smoothness: i.e. for every $g \in \Gamma$, the map $x \mapsto g \cdot x$ is smooth (i.e. C^∞). This is quite natural if we wish the quotient W/Γ to be smooth.

   Freeness: is needed since if we take the quotient of a manifold say by a finite group with fixed points, the quotient is not a manifold (take e.g. the action of $\mathbb{Z}/2\mathbb{Z}$ acting by $x \mapsto -x$ on $\mathbb{R}$. The quotient identifies with $\mathbb{R}_+$ which is not a manifold.

   Properness: this means that the map $(x, g) \mapsto (x, g \cdot x)$ from $W \times \Gamma \to W \times W$ is proper, i.e. that the inverse image of a compact subspace of $W \times W$ in $W \times \Gamma$ is compact. Equivalently: for every compact piece K of W, the set $\{g \in \Gamma;\ K \cap g \cdot K \neq \varnothing\}$ is finite. This condition is needed to ensure that W/Γ is Hausdorff.

   Foliation preserving: means that the image of a leaf is a leaf. Of course, we need this assumption in order to have a foliation in the quotient space.

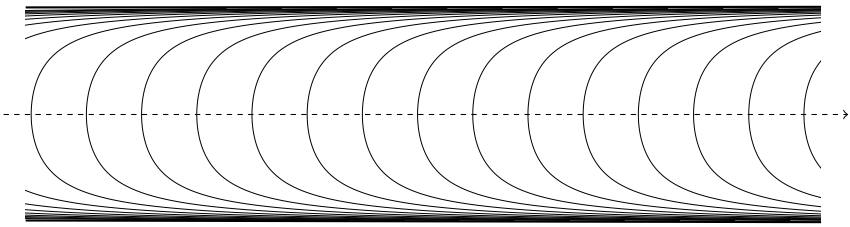
   Let us present typical examples of this situation.

   - Let G be a Lie group, H a connected closed subgroup and Γ a discrete subgroup of G. Let G be foliated by the left action of H on G (i.e. the leaves are the Hx); we then let Γ act on the right: it preserves the foliation, and the quotient G/Γ is foliated. This is also the foliation given by the action of H on the quotient space G/Γ.

As an example, one can take the case of $\mathbb{R}$ sitting in $\mathbb{R}^2$ via the map $t \mapsto (t, t\theta)$ where θ is an irrational number. Taking $\Gamma = \mathbb{Z}^2 \subset \mathbb{R}^2$, we obtain the famous Kronecker foliation.

- Let Q be a compact manifold and let $\Gamma = \pi_1(Q)$ be its fundamental group. Denote by $\widetilde{Q}$ the universal cover. Then Γ acts freely and properly on $\widetilde{Q}$ with quotient Q. Let now Γ act on a (compact) manifold V smoothly—but of course in general neither and properly nor freely. Put $W = \widetilde{Q} \times V$ foliated by $\widetilde{Q} \times \{x\}$ for $x \in V$. Let Γ act on W diagonally. Then $M = W/\Gamma$ is foliated. Note that we have a natural fibration $W/\Gamma \to Q = \widetilde{Q}/\Gamma$ (with fibers V) and that the leaves of the foliation of M are transverse to this fibration. This construction is called a *foliated bundle.*

2. ***Gluing:*** Let M_1, M_2 be manifolds with boundary. Assume that the boundaries of M_1 and M_2 are diffeomorphic. Given foliations $\mathcal{F}_1$ on M_1 and $\mathcal{F}_2$ on M_2 with leaves that are tangent to the boundary and agree on this common boundary. Then, we can glue $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ and obtain a foliation on a manifold without boundary.
   We illustrate this with the famous *Reeb foliation.*
   Start with the pattern of a foliation of $\mathbb{R} \times [-1, 1]$ which is symmetric with respect to the x axes and translation invariant along this axes as shown in the picture below.



PATTERN FOR THE REEB FOLIATION

   Rotate this picture around the axis, so to obtain a foliation of the cylinder $\mathbb{R} \times D$ where D is the unit disc, which is translation invariant.
   Take then the quotient of this by the action of $\mathbb{Z}$. In this way, we obtain a foliation of the solid torus $\mathbb{S}^1 \times D$—where $\mathbb{S}^1$ is the circle, called the *Reeb foliation* of the solid torus. The leaves are two dimensional and the boundary $\mathbb{S}^1 \times \mathbb{S}^1$ is a leaf.
   Now, the sphere $\mathbb{S}^3 = \{(u, v) \in \mathbb{C}^2;\ |u|^2 + |v|^2 = 1\}$ can be decomposed into two manifolds M_1 and M_2 with a common boundary:

$$M_1 = \{(u, v) \in \mathbb{C}^2;\ |u|^2 + |v|^2 = 1,\ |u| \geq |v|\}$$
$$M_2 = \{(u, v) \in \mathbb{C}^2;\ |u|^2 + |v|^2 = 1,\ |u| \leq |v|\}.$$

   The map $(u, v) \mapsto \left(\frac{u}{|u|}, \sqrt{2}\, v\right)$ is a diffeomorphism from M_1 to $\mathbb{S}^1 \times D$. In the same way, M_2 is diffeomorphic to the solid torus. Gluing their Reeb foliations, we obtain a Reeb foliation of the sphere $\mathbb{S}^3$.

Remark [⋆⋆] Let me however note that, if we wish to obtain a smooth foliation of $\mathbb{S}^3$, the pattern above has to be infinitely tangent to the axes $y = 1$ and $y = -1$. The curve passing through the point $(a, 0)$ would be given by an equation of the form $y^2 = 1 - e^{a-x}$.

3. ***Pull back foliations.*** Let $(M, \mathcal{F})$ be a foliated manifold and let $f : N \to M$ be a smooth map. Assume that f is *transverse to the leaves of* $\mathcal{F}$, i.e. that for every $x \in N$, we have $df_x(T_xN) + F_{f(x)} = T_{f(x)}M$—where $F_y \subset T_yM$ is the tangent space to the leaf through y. Then, the sets $f^{-1}(L)$ where L runs over the leaves of $\mathcal{F}$ form a partition of N into submanifolds. Their connected components form a foliation of N denoted by $f^{-1}(\mathcal{F})$.

3.1.3 The Frobenius Theorem

Let M be a smooth manifold of dimension n, and let F be a (smooth) sub vector bundle of rank p of the tangent bundle TM. The sub bundle F is said to be *integrable* or *involutive* if the Lie bracket $[X, Y]$ of any two smooth sections X, Y of F (which are in particular vector fields on M) is a section of F.

Let $(M, \mathcal{F})$ be a foliated manifold. The vectors tangent to the leaf form a subbundle $T_{\mathcal{F}}(M)$ of the tangent bundle. In local coordinates, writing $M \simeq U \times T$ as above, $T_{\mathcal{F}}(M)$ is the set of vectors of the form $(\xi, 0) \in \mathbb{R}^p \times \mathbb{R}^q$.

Theorem 15 (The Frobenius integrability theorem)

1. *Let $(M, \mathcal{F})$ be a foliated manifold. The subbundle $T_{\mathcal{F}}(M)$ of TM consisting of vectors tangent to the leaves is integrable.*
2. *Every integrable subbundle integrates to a foliation: if $F \subset TM$ is an integrable subbundle, there exists a unique foliation $\mathcal{F}$ such that $T_{\mathcal{F}}(M) = F$.*

Proof 1. We may argue locally: we just need to prove that for every open subset $\Omega \subset \mathbb{R}^n$, the subbundle $F = \Omega \times (\mathbb{R}^p \times \{0\})$ of $TU = \Omega \times (R^p \times \mathbb{R}^q)$ is integrable.
Let $\partial_i = \partial/\partial_{t_i}$ be the canonical constant frame for F. We have $[f\partial_i, g\partial_j] = (f\partial_i(g))\partial_j - (g\partial_j(f))\partial_i$. Let X, Y be sections of F, then $X = \sum_{i=1}^{p} f_i\partial_i$ and $Y = \sum_{i=1}^{p} g_i\partial_i$, and it follows that $[X, Y] = XY - YX = \sum_{i,j\le p} [f_i\partial_i, g_j\partial_j] = \sum_{i,j\le p} \Big(f_i(\partial_i(g_j)\partial_j - g_j(\partial_j(f_i)\partial_i\Big)$, which again is a section of F.

2. Assume that F is integrable, and fix $x_0 \in M$. Let $\varphi : M \to \mathbb{R}^p$ be a smooth map such that $\varphi(x_0) = 0$, and $d\varphi_{x_0}$ induces an isomorphism $F_{x_0} \to \mathbb{R}_p$. The set of the points $m \in M$ such that $d\varphi_x$ induces an isomorphism $F_x \to \mathbb{R}^p$ is an open subset Ω_0 of M.
We define the $(X_1, \ldots, X_p)$ sections of F on Ω_0 by the requirement that $d\varphi(X_j) = \partial_j$.

For $i \in \{1, \ldots, p\}$ denote by ψ_i^t the associated flow, defined in a common open neighborhood W of $\Omega_0 \times \{0\}$ in $\Omega_0 \times \mathbb{R}$ (i.e. the solution of the differential equation $d/dt(\psi_i^t(x)) = X_i(\psi_i^t(x))$).
Since F is integrable, there exist smooth functions f_{ij}^k on Ω_0 such that $[X_i, X_j] = \sum_{k=1}^{p} f_{ij}^k X_k$, for all $1 \leq i, j \leq p$. Applying $d\varphi$ to this expression, we see that

$$0 = [\partial_i, \partial_j] = [d\varphi(X_i), d\varphi(X_j)] = d\varphi([X_i, X_j]) = \sum_{k=1}^{p} f_{ij}^k \partial_k$$

whence $f_{ij}^k = 0$ and $[X_i, X_j] = 0$.
Put $T_0 = \varphi^{-1}(\{0\}) \cap \Omega_0$. As the restriction of φ to Ω_0 is a submersion, it is a submanifold of M of dimension q.
Then we can construct

- an open neighborhood T of 0 in $\mathbb{R}^q$ and a diffeomorphism χ from T to an open neighborhood of x_0 in T_0
- $\varepsilon > 0$ such that, for $(t_1, \ldots, t_p)$ in the open ball B_ε of radius ε in the (e.g. euclidean) $\mathbb{R}^n$, the map $\exp_{t_1 X_1} \ldots \exp_{t_p X_p}$ is defined on T_0.

This gives a chart $B_\varepsilon \times T_0$ for the foliation.

□

3.2 Holonomy

Here is a very interesting phenomenon arising on foliations: following a leaf along a path, yields to maps of the transversals. We will use this holonomy in order to construct a natural Lie groupoid associated to the foliation (cf. [7, 29]).

3.2.1 A Smooth Structure on the Graph of a Foliation?

Let R denote the set of pairs of points in the same leaf, i.e. the graph of the equivalence relation associated with the foliation. Being an equivalence relation, R is a groupoid. In order to apply our constructions we would like it to be a Lie groupoid.

Question What is the natural smooth structure on R?

The answer to this question forces us to enlarge a little bit the graph R. We will construct a natural Lie groupoid G and a surjective groupoid morphism $G \to R$. We construct the manifold G by gluing together open subsets of $\mathbb{R}^{2p+q}$ (using smooth maps).

Let us come back to the question of the smooth structure of R. Near a point (x, x) there is a natural answer: write $\Omega \simeq U \times T$. A neighbourhood of (x, x) is the set $\{((u, t), (u', t));\ (u, u', t) \in U \times U \times T\}$.

Take now two charts $\Omega \simeq U \times T$ and $\Omega' \simeq U' \times T'$ and $x \in \Omega$, $y \in \Omega'$ in the same leaf. We want to define a neighbourhood of $(x, y) \in R$ of dimension $2p + q$. So, we need to find a natural way to relate T and T' at least near a pair (x, y) of points of $\Omega \times \Omega'$ which sit in the same leaf.

Definition 18 A *holonomy* is a smooth map $h : T \to T'$ where T, T' are local transversals such that, for all $x \in T$, the elements x and $h(x)$ are in the same leaf.

With this definition in hand, we will now be able to construct our Lie groupoid.

3.2.2 Construction of the Holonomy Groupoid

Let $(\Omega_i, \varphi_i : \Omega_i \to U_i \times T_i)$ be foliation charts defining the foliation, with chart changes $(u, t) \mapsto (g_{i,j}(u, t), h_{i,j}(t))$. Write also $\varphi_i(x) = (\omega_i(x), \tau_i(x))$.

The open cover of the graph. We cover the *graph of the foliation* with sets indexed by triples (i, j, h), where $h : T'_h \to T_h$ is a holonomy from an open subset T'_h of T_j to an open subset T_h of T_i. Put $W_{(i,j,h)} = U_i \times U_j \times T'_h$.

Range and source. Define the maps $s, r : W_{(i,j,h)} \to M$ by setting $s(u, u', t) = \varphi_j^{-1}(u', t)$ and $r(u, u', t) = \varphi_i^{-1}(u, h(t))$.

Composition. Let $W_{i,j,h}$ and $W_{j,k,h'}$ be two charts. The elements $(u, u', t) \in W_{i,j,h}$ and $(u'_1, u'', t') \in W_{j,k,h'}$ are composable if $u'_1 = u'$ ant $t = h'(t')$. Their composition is then $(u, u'', t') \in W_{i,k,h\circ h'}$.

Holonomy germs. Fix two holonomies (i_1, j_1, h_1) and (i_2, j_2, h_2). Let $(x, y) \in (r, s)(W_{(i_1,j_1,h_1)}) \cap (r, s)(W_{(i_2,j_2,h_2)})$. Write $j_2(y) = (u, t)$. We say that the *holonomy germs* of h_1 and h_2 at y agree if $h_1 \circ h_{(j_1,j_2)}$ and $h_{i_1,i_2} \circ h_2$ coincide in a neighborhood of t.

Definition 19 This defines an equivalence relation on $\bigcup_{(i,j,h)} W_{(i,j,h)}$. The quotient is the *graph* of the foliation.

Almost by definition, the graph is

- a smooth manifold $Hol(M, \mathcal{F})$—we just need to notice that
  - the set of $t \in T_j$ such that the germ of two holonomies h and h' coincide is open—just by the definition of a germ
  - the chart changes between two $W_{(i,j,h)}$ are smooth—since they are constructed using the smooth maps φ_i;
- a Lie groupoid. Its algebroid is a sub-algebroid of the algebroid TM of $M \times M$: it is the subbundle $F \subset TM$ of vectors tangent to the leaves of the foliation.

Proposition 6 *The image $(r, s)(Hol(M, \mathcal{F}))$ is the set of pairs of elements in the same leaf.*

Proof $(r, s)(Hol(M, \mathcal{F}))$ defines an equivalence relation in M.

Given x, the class of x is the set $\{r(w);\ w \in Hol(M, \mathcal{F}),\ s(w) = x\}$. It is open in the leaf through x. We thus obtain a partition of every leaf into open subsets. By connectedness the set $\{r(w);\ w \in Hol(M, \mathcal{F}),\ s(w) = x\}$ is the leaf through x. □

3.2.3 Holonomy of a Path

We prove this proposition with a different point of view. Let us first note that if $\varphi_i : \Omega_i \to U_i \times T_i$ and $\varphi_j : \Omega_j \to U_j \times T_j$ are compatible charts and $\varphi_j \circ \varphi_i^{-1}(u, t) = (g_{i,j}(u, t), h_{i,j}(t))$, the map $h_{i,j} : T_i' \to T_j'$ is a holonomy (where T_i' and T_j' are open subsets of the transversals T_i and T_j).

Now a leaf is path connected. Let x, y be two points in the same leaf L and let $\gamma : [0, 1] \to L$ be a continuous path joining x to y, i.e. such that $\gamma(0) = x$ and $\gamma(1) = y$. Covering $[0, 1]$ by finitely many open intervals $I_1, \ldots, I_n$ such that $\gamma(I_j)$ is contained in a chart Ω_{i_j}, $0 \in I_0$ and $1 \in I_n$ and $I_j \cap I_{j+1} \neq \emptyset$, we obtain a holonomy as a composition $h_{i_n,i_{n-1}} \circ h_{i_{n-1},i_{n-2}} \circ \ldots \circ h_{i_2,i_1} : T_{i_1}' \to T_{i_n}'$ where T_{i_1}' is an open subset of the transversal T_{i_1} through x and T_{i_n}' is an open subset of the transversal T_{i_n} through y. This composition maps the class of x in the transversal T_{i_1}' to the class of y in the transversal T_{i_n}'.

Let us make a few remarks on this construction.

- The construction shows that a path in a leaf defines a holonomy.
- By definition, the holonomy of a path only depends on the sequence of charts covering it. In particular, two nearby paths define the same holonomy. One deduces that two homotopic paths define the same holonomy.
- The set of germs of holonomy corresponding to paths is easily seen to form an open subgroupoid of the groupoid $Hol(M, \mathcal{F})$. It is the smallest open subgroupoid of $Hol(M, \mathcal{F})$.

3.3 Topological Index; Index Theorem for Foliations

The statement of any index theorem is always the same:

Theorem 16 $\mathrm{ind}_a(D) = \mathrm{ind}_t([\sigma(D)])$*: the analytic index of a pseudodifferential operator is equal to the topological index of its symbol.*

We already discussed the analytic index of an operator—in the case of a Lie groupoid—and in fact saw that this index only depends on the K-theory class of its symbol.

In this section we discuss cases where a topological index can be defined and an index theorem proved.

3.3.1 The Topological Index

On the Thom isomorphism

In order to define the topological index we need to use the following form of the Thom isomorphism.

Proposition 7 *Let M be a (locally) compact space and let $E, F \to M$ be vector bundles such that $E \oplus F$ is a trivial bundle of even dimension. Then, we have a canonical isomorphism $K^*(E) \to K^*(F)$.*

Proof The construction is the following: use the Bott isomorphism to write $K^*(E) \simeq K^*(E \times \mathbb{R}^{2k}) = K^*(E \oplus E \oplus F)$ and the Thom isomorphism to write $K^*(F) = K^*(E \oplus E \oplus F)$ □

The Topological Index on a Manifold

Let us describe now the topological index in K-theory as defined by Atiyah-Singer in the case of an elliptic operator acting on a manifold—i.e. from the point of view of groupoids, for the groupoid $M \times M$.

Let $i : M \to \mathbb{R}^n$ be a smooth injective immersion—with n even. We have a natural tubular neighborhood construction (Sect. 2.1.3) which is a diffeomorphism of the total space of the normal bundle $N = N_M^{\mathbb{R}^n}$ of the inclusion of M in $\mathbb{R}^n$ into an open neighborhood U of $i(M)$ in $\mathbb{R}^n$. Then the algebra $C_0(U)$ of continuous functions that vanish at infinity on Y identifies with the subalgebra of $C_0(\mathbb{R}^n)$ of continuous functions on $\mathbb{R}^n$ that vanish outside U. We thus get an map $e : K^0(U) = K_0(C_0(U)) \longrightarrow K_0(C_0(\mathbb{R}^n)) = K^0(\mathbb{R}^n)$ (called the excision map).

Remark 8 It follows from Remark 4 that this "excision map" $e : K_0(C^0(N)) \to K_0(\mathbb{R}^n)$ corresponds to the map induced by the deformation groupoid $DNC(\mathbb{R}^n, i(M))$ (which here is just a space) associated with the inclusion $i : M \to \mathbb{R}^n$.

The direct sum of bundles $N \oplus TM$ is the restriction to M of the tangent bundle to $\mathbb{R}^n$, and we therefore have a Thom isomorphism $K^0(T^*M) \to K^0(N)$.

Definition 20 The topological index is the map $K^0(T^*M) \to \mathbb{Z}$ defined as the composition

$$K^0(T^*M) \xrightarrow{\text{Thom}} K^0(N) \simeq K^0(U) \xrightarrow{e} K^0(\mathbb{R}^n) \xrightarrow{\text{Bott}^{-1}} K^0(pt) = \mathbb{Z}.$$

Theorem 17 (Atiyah-Singer index theorem [3]) *Let D be an elliptic (pseudo)-differential operator on M and $[\sigma(D)] \in K^0(T^*M)$ the K-theory class of its symbol. We have* $\text{ind}_a(D) = \text{ind}_t([\sigma(D)])$.

3.3.2 Generalizations

The Atiyah-Singer index theorem was a very deep accomplishment, recovering several previous results as the Riemann–Roch theorem, the Hirzbruch theorem for the signature operator, etc. It introduced many new ideas around pseudodifferential calculus, K-theory. It had several important generalisations.

We will outline here two of them.

Families of Elliptic Operators

In [4], Atiyah and Singer consider the case where we have a family of elliptic operators: consider a fibration $\pi : M \to B$ and operators which only differentiate along the fibers of this fibration.

This corresponds to the subgroupoid $M \times_B M = \{(x, y) \in M \times M;\ \pi(x) = \pi(y)\}$.

An elliptic operator for this groupoid is a family $(P_b)_{b\in B}$, indexed by B, of elliptic operators: for $b \in B$, the operator P_b is a pseudo-differential operator on the fiber on the fiber $F_b = \pi^{-1}(b)$, elliptic along this fiber. By continuity of $b \mapsto P_b$ (or $b \mapsto \sigma(P_b)$, the index is (locally) constant. It can in fact be organized as a difference of vector bundles: $\text{ind}_a(P) \in K^0(B) = K_0(C(B))$.

The symbol of P is a function on the cotangent bundle to the fibers: the dual of $T_F M$: for $x \in M$, $(T_F M)_x$ is the kernel of $(d\pi)_x : T_x M \to T_{\pi(x)} B$.

To define the topological index in this setting, we use an embedding $i : M \to \mathbb{R}^n \times B$ of the form $i(x) = (j(x), \pi(x))$.

Denote by $N = N_M^{B\times\mathbb{R}^n}$ the normal bundle of the embedding i. Note that, for every $x \in M$, we have $T_{i(x)}(\mathbb{R}^n \times B) = \mathbb{R}^n \times T_{\pi(x)} B$. As $d\pi_x$ is surjective, the quotient $N_x = T_{i(x)}(\mathbb{R}^n \times B)/di(T_x M)$ identifies with the quotient $\mathbb{R}^n/dj(T_F M)_x$. Therefore the bundle $T_F M \oplus N_M^{B\times\mathbb{R}^n}$ identifies with the trivial bundle $M \times \mathbb{R}^n$.

Definition 21 The topological index is the map $K^0(T^*_F M) \to K^0(B)$ defined as the composition

$$K^0(T^*_F M) \xrightarrow{\text{Thom}} K^0(N) \simeq K^0(U) \xrightarrow{e} K^0(B \otimes \mathbb{R}^n) \xrightarrow{\text{Bott}^{-1}} K^0(B).$$

Here U is a tubular neighborhood of $i(M)$ in $B \times \mathbb{R}^n$.

Theorem 18 (Atiyah-Singer index theorem for families [4]) *The topological and analytic index coincide in this setting.*

As $K^1(B)$ is not necessarily trivial, there is also a topological and analytical index $K^1(T^*_F M) \to K^1(B)$ (which also coincide).

The Foliation Case

Let $(M, \mathcal{F})$ be a foliated manifold. Denote by $Hol(M, \mathcal{F})$ its holonomy groupoid and $C^*(M, \mathcal{F}) = C^*(Hol(M, \mathcal{F}))$ its C^*-algebra. As the algebroid of $Hol(M, \mathcal{F})$ identifies with the subbundle $F \subset TM$ of vectors tangent to the leaves, the analytic index is a map $K^i(F^*) \to K_i(C^*(M, F))$ $(i = 0, 1)$.

A difficulty that one encounters when trying to define the topological index $K^i(F^*) \to K_i(C^*(M, F))$ comes form the fact that it is not clear what could be a topological cycle for $(M, \mathcal{F})$ i.e. a topologically defined element of K-element of $K_i(C^*(M, \mathcal{F}))$.

The answer to that problem as given by Alain Connes is the following: let T be a (locally closed) transversal of a foliated manifold (M, F). In other words, T is a (locally closed) submanifold of M and, for $x \in T$, we have $T_x M = T_x T \oplus F_x$. The normal bundle N_T^M of T in M identifies with the restriction to T of the subbundle $F \subset TM$.

A small (tubular) neighborhood of $T \subset M$ is then obtained by exponentiation F, i.e. along the leaves. We therefore have an open subset $\Omega \subset M$ which is diffeomorphic to the total space of the normal bundle N_T^M. The restriction of $\mathcal{F}$ to M is then the foliation given by the fibers of the projection $N_T^M \to T$, and $C^*(\Omega, \mathcal{F}_\Omega)$ is isomorphic to $C_0(T) \otimes \mathcal{K}$ into which $C_0(T)$ embeds using a rank one projection of $\mathcal{K}$.

In this, way we obtain an inclusion $C_0(T) \subset C^*(\Omega, \mathcal{F}_\Omega) \subset C^*(M, \mathcal{F})$, and thus a morphism

$$K^*(T) \xrightarrow{\hat{e}} K_*(C^*(M, \mathcal{F})).$$

Images of morphisms $\hat{e} : K^*(T) \to K_*(C^*(M, \mathcal{F}))$ where T is a transversal will be considered as *topological cycles.*

Let us now define the topological index. Let $i : M \to \mathbb{R}^n$ be an embedding—with n even. Denote by $(M \times \mathbb{R}^n, \mathcal{F} \otimes \{pt\})$ the foliation of $M \times \mathbb{R}^n$ whose leaves are $L \times \{t\}$ for L leaf of $(M, \mathcal{F})$ and $t \in \mathbb{R}^n$.

Let N be the normal bundle to $di(F)$: for $x \in M$, we define N_x the orthogonal in $\mathbb{R}^n$ of $(di)_x(F_x)$.

The map $j : N \to M \times \mathbb{R}^n$ given by $j(x, \xi) = (x, i(x) + \xi)$ is an embedding which identifies (the total space of) N with the closed submanifold $W = \{(x, y) \in M \times \mathbb{R}^n;\ y - i(x) \in N_x\}$ of $M \times \mathbb{R}^n$.

Let $(x, y) \in W$; put $\xi = y - i(x) \in N_x$ so that $(x, y) = j(x, \xi)$, and

$$\begin{aligned} T_{(x,y)}W = dj(T_{(x,\xi)}N) &= \{(X, (di)_x(X) + Y);\ X \in T_xM,\ Y \in N_x\} \\ &= \{(X, Z) \in T_xM \times \mathbb{R}^n;\ Z - (di)_x(X) \in N_x\}. \end{aligned}$$

It follows that $F_x \times \{0\} \oplus T_{(x,y)}W = T_xM \times \mathbb{R}^n$. In other words, W is a transversal of the foliation $(M \times \mathbb{R}^n, \mathcal{F} \otimes \{pt\})$.

Definition 22 (Connes) The topological index is the composition

$$K^*(F^*) \xrightarrow{\text{Thom}} K^*(N) \simeq K^*(W) \xrightarrow{\hat{e}} K(C^*(M \times \mathbb{R}^n, \mathcal{F} \times \{pt\})) \xrightarrow{\text{Bott}^{-1}} K(C^*(M, \mathcal{F})).$$

We used the isomorphism $C^*(M \times \mathbb{R}^n, \mathcal{F} \otimes \{pt\}) = C^*(M, \mathcal{F}) \otimes C_0(\mathbb{R}^n)$.

Theorem 19 (Longitudinal index theorem for foliations [9]) $\text{ind}_a = \text{ind}_t$.

3.3.3 On the Proofs

Let us outline here a proof of the longitudinal index theorem for foliations (Theorem 19) based on deformation groupoids.

Proof Let $i : M \to \mathbb{R}^n$ be an embedding, with n even, the groupoid $G = Hol(M, \mathcal{F}) \times \mathbb{R}^n \rightrightarrows M \times \mathbb{R}^n$, which is the product of the holonomy groupoid $Hol(M, \mathcal{F}) \rightrightarrows M$ with the space $\mathbb{R}^n$ considered as a groupoid $\mathbb{R}^n \overset{\text{id,id}}{\rightrightarrows} \mathbb{R}^n$.

The submanifolds $M_0 = M \times \{0\}$ and $M_1 = \{(x, i(x));\ x \in M\}$ of $M \times \mathbb{R}^n = G^{(0)}$ will be considered as subgroupoids of G.

The groupoids $DNC(G, M_i)$ give rise to K-theory maps $\varphi_i : K_0(C^*(N_{M_i}^G)) \to K_0(C^*(G))$ (for $i = 0, 1$). Note that $C^*(G) = C^*(M, \mathcal{F}) \otimes C_0(\mathbb{R}^n)$ so that $K_0(C^*(G))$ identifies with $C^*(Hol(M, \mathcal{F}))$ (using Bott periodicity).

$(*)$ Loosely speaking, the map φ_0 corresponds to the analytic index of $(M, \mathcal{F})$ and the map φ_1 to the topologic one.

$(*)$ Since the inclusions $x \mapsto (x, 0)$ and $x \mapsto (x, i(x))$ from M to $M \times \mathbb{R}^n$ are homotopic, it follows that these maps coincide. □

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Lectures on the Euler Characteristic of Affine Manifolds

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Abstract These lecture notes are the result of a minicourse given at the summer school *Geometric, algebraic and topological methods in quantum field theory*, held in Villa de Leyva in July 2017. Our goal is to provide an introduction to a conjecture of Chern that states that the Euler characteristic of a closed affine manifold vanishes. We present part of the history and motivation for the conjecture as well as some recent developments.

1 Introduction

An affine structure on a manifold is an atlas whose transition functions are affine transformations. The existence of such a structure is equivalent to the existence of a flat torsion free connection on the tangent bundle. Chern's conjecture states the following:

Conjecture (Chern ∼ 1955)
The Euler characteristic of a closed affine manifold is zero.

In case the connection ∇ is the Levi–Civita connection of a riemannian metric, the Chern–Gauss–Bonnet formula:

$$\chi(M) = \left(\frac{1}{2\pi}\right)^n \int_M \mathsf{Pf}(K)$$

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implies that the Euler characteristic is zero. However, not all flat torsion free connections on TM admit a compatible metric, and therefore, Chern–Weil theory cannot be used in general to write down the Euler class in terms of the curvature.

This chapter contains an exposition of the following results concerning the Euler characteristic of affine manifolds:

- In 1955, Benzécri [1] proved that a closed affine surface has zero Euler characteristic.
- In 1958, Milnor [13] proved inequalities which completely characterise those oriented rank two bundles over a surface that admit a flat connection.
- In 1975, Kostant and Sullivan [11] proved Chern's conjecture in the case where the manifold is complete.
- In 1977, Smillie [17] proved that the condition that the connection is torsion free matters. For each even dimension greater than 2, Smillie constructed closed manifolds with non-zero Euler characteristic that admit a flat connection on their tangent bundle.
- In 2015, Klingler [16] proved the conjecture for special affine manifolds. That is, affine manifolds that admit a parallel volume form.

As far as we can tell, the general case of Chern's conjecture remains open.

We have added appendices with preliminary material. Appendix A contains a review of the theory of connections, curvature and the Chern–Gauss–Bonnet theorem. Appendix B contains an introduction to spectral sequences and Appendix C to sheaf theory.

1.1 Affine Manifolds

Definition 1 A diffeomorphism φ between open subsets of $\mathbb{R}^m$ is affine if it has the form

$$\varphi(x) = Ax + b,$$

where $A \in \mathrm{GL}(m, \mathbb{R})$ and $b \in \mathbb{R}^m$.

Definition 2 An affine structure on a manifold is an atlas such that all transition functions are affine and it is maximal with this property. An affine manifold is a manifold together with an affine structure.

It is possible to characterise affine structures in a more intrinsic manner that can be expressed without reference to an atlas, as the following lemma shows:

Lemma 1 *Let M be a manifold. There is a natural bijective correspondence between affine structures on M and flat torsion free connections on TM.*

Proof Let $(U_\alpha, \varphi_\alpha)$ be an affine structure on M. There is a unique connection ∇ on TM whose Christoffel symbols vanish in affine coordinates. Conversely, given a flat torsion free connection ∇ on TM, the set of coordinates for which the Christoffel symbols of ∇ vanish gives an affine structure on M. □

Example 1 The torus $\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m$ has a natural affine structure for which the projection map $\pi : \mathbb{R}^m \to \mathbb{T}^m$ is an affine local difeomorphism.

Example 2 (*Hopf manifolds*) Let us fix a real number $\lambda > 1$ and consider the action of the group $\mathbb{Z}$ on $\mathbb{R}^m - \{0\}$ given by:

$$n \star x := \lambda^n x.$$

Since the action is free and proper, the quotient is a smooth manifold called the Hopf manifold Hopf^m_λ. Since the group $\mathbb{Z}$ acts by affine transformations the quotient space is an affine manifold. Topologically, these manifolds are the union of two circles for $m = 1$ and diffeomorphic to $S^{m-1} \times S^1$ for $m > 1$.

Definition 3 An affine structure on a Lie group G is called left invariant if for all $g \in G$:

$$(L_g)^*(\nabla) = \nabla,$$

where L_g denotes the diffeomorphism given by left multiplication by g.

Definition 4 Let V be a real vector space. A bilinear map:

$$\beta : V \otimes V \to V;\ \ v \otimes w \mapsto v \cdot w$$

is called left symmetric if:

$$v \cdot (w \cdot z) - (v \cdot w) \cdot z = w \cdot (v \cdot z) - (w \cdot v) \cdot z.$$

Definition 5 An affine structure on a finite dimensional real Lie algebra $\mathfrak{g}$ is a left symmetric bilinear form on $\mathfrak{g}$ such that for all $v, w \in \mathfrak{g}$:

$$[v, w] = v \cdot w - w \cdot v.$$

Lemma 2 *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a natural bijective correspondence between left invariant affine structures on G and affine structures on $\mathfrak{g}$.*

Proof For any $v \in \mathfrak{g} = T_eG$ denote by $\widehat{v}$ the corresponding left invariant vector field. Given a left invariant affine structure on G we define a bilinear form on $\mathfrak{g}$ by:

$$v \cdot w := \nabla_{\widehat{v}} \widehat{w}(e).$$

Since ∇ is torsion free we know that:

$$v \cdot w - w \cdot v = \nabla_{\widehat{v}}\widehat{w}(e) - \nabla_{\widehat{w}}\widehat{v}(e) = [v, w].$$

Using the fact that ∇ is flat and left invariant, we compute:

$$\begin{aligned} v \cdot (w \cdot z) - (v \cdot w) \cdot z - w \cdot (v \cdot z) + (w \cdot v) \cdot z \\ = \nabla_{\widehat{v}}(\nabla_{\widehat{w}}(\widehat{z})) - \nabla_{\nabla_{\widehat{v}}(\widehat{w})}(\widehat{z}) - \nabla_{\widehat{w}}(\nabla_{\widehat{v}}(\widehat{z})) + \nabla_{\nabla_{\widehat{w}}(\widehat{v})}(\widehat{z}) \\ = \nabla_{[\widehat{v},\widehat{w}]}(\widehat{z}) - \nabla_{[\widehat{v},\widehat{w}]}(\widehat{z}) \\ = 0. \end{aligned}$$

We conclude that the bilinear form is left symmetric. Conversely, given an affine structure on $\mathfrak{g}$ there is a unique left invariant connection ∇ such that:

$$\nabla_{\widehat{v}}\widehat{w}(e) = v \cdot w.$$

The computations above show that this connection is flat and torsion free. □

Example 3 The Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ admits a natural affine structure given by:

$$A \cdot B := AB.$$

We conclude that the Lie group $\mathrm{GL}(n, \mathbb{R})$ admits a left invariant affine structure.

The question of which Lie groups admit left invariant affine structures is an interesting problem. In [14], Milnor asked whether solvable Lie algebras admit affine structures. In [4], Burde showed that the answer to this question is negative.

1.2 Complete Manifolds and the Developing Map

In this section we introduce the developing map of a simply connected affine manifold and use it to characterise complete affine manifolds.

Theorem 4 *Let M be an affine manifold of dimension m and G be the group $\mathsf{Aff}(\mathbb{R}^m) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ seen as a discrete group. There is a natural principal G-bundle $\pi : \tau(M) \to M$ such that sections of π are in natural bijective correspondence with affine immersions from M to $\mathbb{R}^m$.*

Proof For each $p \in M$ we define

$$C_p := \left\{\varphi : U \to V \subseteq \mathbb{R}^m \mid \varphi \text{ is an affine diffeomorphism and } p \in U\right\}$$

There is an equivalence relation $\sim$ on C_p given by declaring that $\varphi \sim \varphi'$ if an only if there exists an open subset $W \subseteq U \cap U'$ such that $p \in W$ and $\varphi|_W = \varphi'|_W$. Let us denote by L_p the set of equivalence classes of elements in C_p and set

$$\tau(M) := \coprod_p L_p.$$

There is a natural map: $\pi : \tau(M) \to M$ that sends L_p to p. The Lie group G acts on each set L_p by composition:

$$g \cdot \varphi := g \circ \varphi,$$

and this action is free and transitive. Moreover, an affine chart $\varphi : U \to V \subseteq \mathbb{R}^m$ induces a natural identification:

$$\widehat{\varphi} : \mathsf{Aff}(\mathbb{R}^m) \times U \to \pi^{-1}(U); \ \ (g, p) \mapsto [g \circ \varphi]_p,$$

where $[g \circ \varphi]_p$ denotes the class of $g \circ \varphi$ in L_p. There is a unique topology on $\tau(M)$ such that $\widehat{\varphi}$ is a homeomorphism for all affine diffeomorphisms φ. The map $\pi : \tau(M) \to M$ is a principal G bundle with respect to this topology. Let σ be a section of π. Then we define $\widetilde{\sigma} : M \to \mathbb{R}^m$ by:

$$\widetilde{\sigma}(p) := \sigma(p)(p).$$

By construction, the map $\widetilde{\sigma}$ is an affine immersion. Conversely, an affine immersion $f : M \to \mathbb{R}^m$ defines a section σ_f given by:

$$\sigma_f(p) := [f]_p.$$

□

Corollary 1 *Let M be a simply connected affine manifold. Any affine chart $\varphi : U \to V \subseteq \mathbb{R}^m$ extends uniquely to an affine immersion from M to $\mathbb{R}^m$.*

Proof Since M is simply connected, the covering space $\pi : \tau(M) \to M$ is trivial and therefore any local section extends uniquely to a global one. □

Definition 6 A developing map for an affine simply connected manifold is an affine immersion into $\mathbb{R}^m$.

Corollary 2 *If M is an affine connected manifold with finite fundamental group then M is not compact.*

Proof If M is compact then so is its universal cover $\widetilde{M}$ which is simply connected and therefore admits an immersion to $\mathbb{R}^m$. This is impossible. □

Definition 7 An affine manifold M with affine connection ∇ is called complete if all geodesics can be extended to arbitrary time.

Lemma 3 *Let M and N be connected affine manifolds of the same dimension and $f, g : M \to N$ affine immersions. If f and g coincide on a nonempty open subset of M then they are equal.*

Proof First we observe that the lemma is true if M and N are connected open subsets of $\mathbb{R}^m$. For the general case, let $X \subset M$ be the subset of M that consists of points $p \in M$ such that f and g coincide in an open neighborhood of p. Clearly, X is open. It suffices to show that X is closed. Fix $p \in X$ and $q \in M$ arbitrary. Since M is connected, there is a path $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Set

$$Y := \gamma^{-1}(M - X).$$

We would like to show that $Y = \emptyset$. Suppose that Y is nonempty and set $y := \inf(Y)$ and $z = \gamma(y)$. Choose affine coordinates with connected domains $\varphi : U \to V$ around z and $\psi : W \to Z$ around $f(z)$ such that $f(U) \subseteq W$ and $g(U) \subseteq W$. Since p is in X there exists $l < y$ such that $\gamma(l) \in U \cap X$. Then $f|_U$ and $g|_U$ coincide in an open neighborhood. Since both U and W are isomorphic to open subsets of $\mathbb{R}^m$ we conclude that $f|_U = g|_U$. This is a contradiction. □

Lemma 4 *Let M be a simply connected complete affine manifold of dimension m. Then any affine coordinate $\varphi : U \to V \subseteq \mathbb{R}^m$ can be extended uniquely to a diffeomorphism from M to $\mathbb{R}^m$*

Proof Fix a point $p \in M$ and affine coordinates $\varphi : U \to V$. By Corollary 1, φ can be extended uniquely to an affine immersion $\widetilde{\varphi} : M \to \mathbb{R}^m$. We will prove that $\widetilde{\varphi}$ is a diffeomorphism. Since M is complete, the exponential map is defined on the whole tangent space T_pM:

$$\mathsf{Exp} : T_pM \to M.$$

On the other hand, the derivative of φ^{-1} at $x = \varphi(p)$ is an isomorphism from $\mathbb{R}^m$ to T_pM. We claim that the map:

$$\psi(y) := \mathsf{Exp} \circ D_x(\varphi^{-1})(y - x)$$

is the inverse of $\widetilde{\varphi}$. By Lemma 3, it suffices to show that the functions are inverses to each other in a small open neighborhood. For $q \in U$ we compute:

$$\psi(\widetilde{\varphi}(q)) = \mathsf{Exp} \circ D_x(\varphi^{-1})(\varphi(q) - x) = \varphi^{-1} \circ \mathsf{Exp}(\varphi(q) - x)) = q.$$

Conversely, for $y \in V$ we have:

$$\widetilde{\varphi} \circ \psi(y) = \varphi \circ \mathsf{Exp} \circ D_x(\varphi^{-1})(y - x) = \mathsf{Exp} \circ D\varphi \circ D_x(\varphi^{-1})(y - x) = y.$$

□

Proposition 1 *Let M be an affine manifold. The following statements are equivalent:*

1. *M is complete.*
2. *There is an affine diffeomorphism $M \cong \mathbb{R}^m / \Gamma$, where Γ is a discrete subgroup of $\mathsf{Aff}(\mathbb{R}^m)$ and the projection $\pi : \mathbb{R}^m \to M$ is the universal cover of M.*

Proof Let us assume that M is complete. Then $\widetilde{M}$ is also complete and simply connected. By Proposition 1 there is an affine difeomorphism $\varphi : \widetilde{M} \to \mathbb{R}^m$. Let us fix a point $p \in M$. Then there is an action of $\pi_1(M, p)$ on $\widetilde{M}$ and therefore on $\mathbb{R}^m$. Since this action is by affine transformations, it defines a homomorphism $\rho : \pi_1(M, p) \to \mathsf{Aff}(\mathbb{R}^m)$. If we set $\Gamma := \mathsf{Im}(\rho)$ we get by construction that $M \cong \mathbb{R}^m / \Gamma$. Conversely, any manifold of the form $\mathbb{R}^m / \Gamma$ is complete. □

2 Some Classical Results

2.1 Milnor's Inequalities and the Conjecture in Dimension $d = 2$

Throughout this section we will set $G = \mathrm{GL}^+(2, \mathbb{R})$, $S = \mathrm{SO}(2) = \mathrm{U}(1)$ and Σ will be a closed oriented surface of positive genus g. Oriented rank two vector bundles over Σ are classified by their Euler class. Milnor considered the question of which of those bundles admit a flat connection. The answer is given by his famous inequality.

Definition 8 Let $\pi : E \to \Sigma$ be a rank two oriented vector bundle. The degree of E is the number

$$D(E) = \int_\Sigma e(E).$$

Here $e(E)$ denotes the Euler class of E.

The Theorem we aim to prove is the following

Theorem
A rank two oriented vector bundle $\pi : E \to \Sigma$ admits a flat connection if and only if

$$|D(E)| < g.$$

Milnor's inequality gives, in particular, a positive answer to Chern's conjecture in dimension $d = 2$, a result that had been previously obtained by Benzecri [1].

The classification of rank two oriented bundles is of course the same as the classification of $\mathrm{GL}^+(2, \mathbb{R})$ principal bundles.

Lemma 5 *The inclusion $\iota : S \hookrightarrow G$ is a homotopy equivalence and therefore the classifying spaces BS and BG are homotopy equivalent.*

Proof By the spectral theorem every nonsingular matrix γ can be written uniquely in the form $\gamma = OR$, where O is orthogonal matrix and R is symmetric positive definite matrix. Then the following function defines a retraction:

$$\begin{aligned} \theta : G &\longrightarrow S \\ \gamma &\longmapsto O \end{aligned}$$

□

Exercise 1 Show that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

then

$$\theta(A) = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Here $x = a + d$ and $y = b - c$. Moreover, show that:

1. If $A \in G$ has positive trace, then $\theta(A)$ has positive trace.
2. If A and B are symmetric and positive definite matrices, then AB has positive trace.
3. If $A, B \in G$ are symmetric and positive definite matrices, then $\theta(AB)$ has positive trace.
4. $\theta(A^{-1}) = \theta(A)^{-1}$.

Lemma 6 *Let $\mathsf{Bun}(\Sigma)$ be the set of isomorphism classes of rank two oriented vector bundles over Σ. The degree map*

$$\begin{aligned} D : \mathsf{Bun}(\Sigma) &\longrightarrow \mathbb{Z} \\ E &\longmapsto D(E) \end{aligned}$$

is a bijection.

Proof The set $\mathsf{Bun}(\Sigma)$ is in bijective correspondence with the set of isomorphism classes of principal G bundles over Σ. These are classified by the set $[\Sigma, BG]$ of homotopy classes of maps to BG. Moreover:

$$[\Sigma, BG] \cong [\Sigma, BS] \cong [\Sigma, \mathbb{CP}^\infty] = [\Sigma, K(\mathbb{Z}, 2)] \cong H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.$$

By definition of the Euler class, this bijection is given by the degree. □

Exercise 2 Fix a point $p \in \Sigma$ and a homomorphism $\rho : \pi_1(\Sigma, p) \to G$. The group $\pi = \pi_1(\Sigma, p)$ acts on the right on the universal cover $\widetilde{\Sigma}$ and on the left on $\mathbb{R}^2$ via the representation ρ. Let $\widetilde{\Sigma} \times_\rho \mathbb{R}^2$ be the quotient of $\widetilde{\Sigma} \times \mathbb{R}^2$ via the equivalence relation

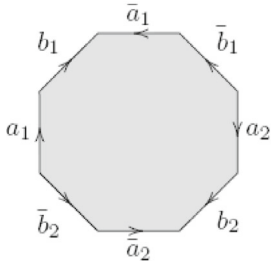
$$(mg, v) \sim (m, \rho(g)v).$$

Show that $\widetilde{\Sigma} \times_\rho \mathbb{R}^2$ has a natural structure of a flat oriented vector bundle over Σ. We will denote this vector bundle E^ρ and call it the bundle associated to ρ.

Lemma 7 *A vector bundle over Σ admits a flat connection if an only if it is isomorphic to one of the form E^ρ.*

Proof By the previous exercise, E^ρ admits a flat connection. Assume that E admits a flat connection ∇ and fix a point $p \in \Sigma$ and an isomorphism $E_p \cong \mathbb{R}^2$. The holonomy of ∇ gives a representation $\rho : \pi_1(\Sigma, p) \to G$ and an isomorphism $E \cong E^\rho$. □

An oriented closed surface Σ of genus g admits a CW structure with one zero-dimensional cell, $2g$ one-dimensional cells and one two-dimensional cell. For example, the surface of genus 2 can be obtained by the following identification:



In general, this decomposition gives the following presentation of the fundamental group of Σ:

$$\pi_1(\Sigma) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

Here $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. Thus, a representation ρ of $\pi_1(\Sigma)$ on G is the same as a choice of matrices $A_1, \ldots, A_g, B_1, \ldots B_g \in G$ such that

$$\prod_{i=1}^{g} [A_i, B_i] = \mathsf{id}.$$

We have seen that such a representation gives rise to a vector bundle E^ρ and the natural question arises of computing the degree of E^ρ in terms of the matrices A_i, B_i.

Let $\widetilde{G}$ be the universal cover of G and consider the short exact sequence

$$0 \to \pi_1(G) \xrightarrow{\iota} \widetilde{G} \xrightarrow{p} G \to 0.$$

Consider also the map $\phi : \mathbb{R} \to S$ given by:

$$\alpha \longmapsto \phi(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

The map ϕ is the universal covering map of S and therefore the retraction map $\theta : G \to S$ lifts naturally to a map:

$$\widetilde{\theta} : \widetilde{G} \to \mathbb{R}.$$

Thus we obtain the following commutative diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & & \pi_1(G) & \longrightarrow & \mathbb{Z} \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle 2\pi} \\ & & \widetilde{G} & \xrightarrow{\widetilde{\theta}} & \mathbb{R} \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \phi} \\ \pi_1(\Sigma) & \xrightarrow{\rho} & G & \xrightarrow{\theta} & S \end{array}$$

Exercise 3 Let G be a connected Lie group with universal cover $p : \widetilde{G} \to G$. Show that if $g \in \widetilde{G}$ is such that $p(g) \in Z(G)$ then $g \in Z(\widetilde{G})$.

Given a homomorphism $\rho : \pi_1(\Sigma) \to G$ determined by matrices $A_1, \ldots, A_g$, $B_1, \ldots, B_g$, we define the number $\delta(\rho)$ by

$$\delta(\rho) := \frac{1}{2\pi}\widetilde{\theta}\left(\prod_{i=1}^{g}[\alpha_i, \beta_i]\right).$$

Here α_i, β_i are elements in $\widetilde{G}$ such that $p(\alpha_i) = A_i$ and $p(\beta_i) = B_i$.

Lemma 8 *The number $\delta(\rho)$ is an integer and depends only on the representation ρ.*

Proof Let us first prove that for any choice of the α_i and β_i the number $\delta(\rho)$ is an integer. By construction, $\prod_{i=1}^{g}[\alpha_i, \beta_i] \in \ker(p)$. This implies that $\widetilde{\theta}(\prod_{i=1}^{g}[\alpha_i, \beta_i])$ is a multiple of 2π and the result follows. Let us now show that the number is well defined. Suppose that α_i' and β_i' is a different choice. Then, for each i we know that:

$$\alpha_i' = \alpha_i x_i \quad \beta_i' = \beta_i y_i,$$

where the x_i and the y_i are in the kernel of p which, by the previous exercise, is contained in the center of $\widetilde{G}$. This implies that:

$$[\alpha_i, \beta_i] = [\alpha_i', \beta_i'],$$

and the result follows. □

Lemma 9 *Given a representation $\rho : \pi_1(\Sigma) \to G$ we have:*

$$\delta(\rho) = D(E^\rho).$$

The idea of the proof is to construct a section of E^ρ with a single zero whose index is precisely $\delta(\rho)$. Hopf's Index Theorem then guarantees that $\delta(\rho) = D(E^\rho)$. The first step towards constructing the section will be to define a path homotopic to the one used in the definition of $\delta(\rho)$. The following two lemmas will serve to prove said homotopy.

Lemma 10 *Let $A, B \in G$ and $\alpha, \beta : [0, 1] \to G$ be paths connecting the identity to A and B respectively. The paths $\alpha\beta$ and $A\beta * \alpha$ connecting the identity to AB are homotopic.*

Proof For every $s \in [0, 1]$ define paths κ_s, σ_s as follows:

$$\kappa_s(t) = \alpha(s(t-1)+1), \qquad \sigma_s(t) = \alpha(t - st).$$

Define the homotopy $H(s, t) = (\kappa_s\beta * \sigma_s)(t)$. Note that $\sigma_s(1) = \alpha(1-s) = \kappa_s(0)\beta(0)$, therefore the concatenation is well defined. One readily verifies that

$$H(0, t) = (A\beta * \alpha)(t), \quad H(1, t) = \alpha(t)\beta(t), \quad H(s, 0) = \mathsf{id}, \quad H(s, 1) = AB.$$

□

Lemma 11 *Fix $v \in \mathbb{R}^2 - \{0\}$. Consider the map $\sigma_v : G \to \mathbb{S}^1$ defined by*

$$\sigma_v(A) = \frac{Av}{||Av||}$$

Then all maps σ_v are homotopic; furthermore, they are all homotopic to the map $\theta : G \to S \cong \mathbb{S}^1$.

Proof For the first part of the lemma take vectors $v, v' \in \mathbb{R}^2 - \{0\}$ and a path $\gamma : [0, 1]$ from v to v' such that $\gamma(t) \neq 0$ for every t. Then $H(A, t) = A\gamma(t)/||A\gamma(t)||$ is a homotopy between σ_v and $\sigma_{v'}$.

For the second part we take the first element of the canonical basis $e = (1, 0)$ and prove that σ_e is homotopic to θ. For this purpose we recall the explicit formula for θ proven in Exercise 1:

$$\theta(A) = \theta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{(a+d)^2 + (b-c)^2}} \begin{bmatrix} a+d & b-c \\ c-b & a+d \end{bmatrix}$$

The isomorphism $S \cong \mathbb{S}^1$ sends a matrix to its first column, therefore θ seen as a map with values in $\mathbb{S}^1$ is given by

$$\theta(A) = \theta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{(a+d)^2 + (b-c)^2}} \begin{bmatrix} a+d \\ c-b \end{bmatrix}$$

On the other hand, the formula for σ_e is

$$\sigma_e(A) = \sigma_e \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{a^2 + c^2}} \begin{bmatrix} a \\ c \end{bmatrix}$$

A homotopy between θ and σ_e is then given by

$$H_t(A) = H_t \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{(a+td)^2 + (tb-c)^2}} \begin{bmatrix} a+td \\ c-tb \end{bmatrix}$$

We check that H_t is well defined. Suppose that $(a+td)^2 + (tb-c)^2 = 0$, then we have $a = -td$ and $c = tb$ for some $t \in [0, 1]$, whence $\det(A) = -t(b^2 + d^2) \leq 0$. □

Now we are ready to prove Lemma 9:

Proof Recall that we wish to construct a section of E^ρ with a single zero with index $\delta(\rho)$.

Consider the path $\gamma : [0, 1] \to G$ given by the formula

$$t \mapsto \gamma(t) = \prod_{i=1}^{g} [\alpha_i(t), \beta_i(t)].$$

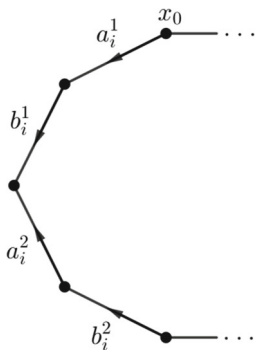
Clearly we have $\gamma(0) = \gamma(1) = \mathsf{id}$ and therefore the composition $\theta \circ \gamma$ induces a map $\widehat{\gamma} : \mathbb{S}^1 \to \mathbb{S}^1$. Let $\widetilde{\gamma} : [0, 1] \to \mathbb{R}$ be the unique lifting of $\theta \circ \gamma$ that begins at 0. Considering the paths $\alpha_i, \beta_i, \gamma$ as elements of $\widetilde{G}$, we see that $\widetilde{\gamma}$ is actually given by the formula

$$\widetilde{\gamma} = \widetilde{\theta}(\gamma) = \widetilde{\theta}\left(\prod_{i=1}^{g} [\alpha_i, \beta_i] \right)$$

Since the degree of $\widehat{\gamma}$ is just its homotopy class, it can be calculated via the lifting $\widetilde{\gamma}$ as follows

$$\deg(\widehat{\gamma}) = \frac{1}{2\pi} \widetilde{\theta}\left(\prod_{i=1}^{g} [\alpha_i, \beta_i] \right) = \delta(\rho).$$

Fig. 1 Polygon



Now let us define the section. If Σ has genus g, then it may be obtained from a polygon with $4g$ sides glued together as follows: suppose that we have a portion of the polygon as shown in Fig. 1, then the sides are glued following the rule

$$a_i^1(t) \sim a_i^2(t), \qquad b_i^1(t) \sim b_i^2(t),$$

for $i = 1, \cdots, g$. Notice that sides a_i^1, a_i^2 map to the generator $[a_i]$ of $\pi_1(\Sigma)$. Similarly, paths b_i^1, b_i^2 map to generator $[b_i]$. We first define the section $s : \Sigma \to \widetilde{\Sigma} \times \mathbb{R}^2$ in the boundary of the polygon using the liftings α_i, β_i. Suppose that we have $s(x_0) = ([x_0], v)$, where $[x_0] \in \widetilde{\Sigma}$ denotes the class of the constant path. Then we define s by

$$\begin{aligned}
s(a_i^1(t)) &= ([a_{i,t}], \alpha_i^{-1}(t)v) \\
s(b_i^1(t)) &= ([b_{i,t} * a_i], A_i^{-1}\beta_i^{-1}(t)v) \\
s(a_i^2(t)) &= ([a_{i,t} * \bar{a}_i * b_i * a_i], A_i^{-1}B_i^{-1}A_i\alpha_i^{-1}(t)v) \\
s(b_i^2(t)) &= ([b_{i,t} * \bar{b}_i * \bar{a}_i * b_i * a_i], A_i^{-1}B_i^{-1}A_iB_i\beta_i^{-1}(t)v).
\end{aligned}$$

In the previous equations, $[a_{i,t}]$ and $[b_{i,t}]$ denote the classes of the paths running from $a_i(0)$ to $a_i(t)$ and from $b_i(0)$ to $b_i(t)$ respectively. Notice that

$$s(a_i^1(t)) \sim s(a_i^2(t)), \qquad s(b_i^1(t)) \sim s(b_i^2(t)),$$

which means that the section is well defined after gluing the sides of the polygon. Furthermore, if $v \neq 0$ then s is non-zero. Next we extend the section to the whole polygon by identifying it with the unitary disk $\mathbb{D}$. Then s may be extended to the interior of $\mathbb{D}$ by

$$s(x) = \begin{cases} ||x|| s\left(\frac{x}{||x||}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The index of the single zero of this section is the degree of the map obtained by multiplying a closed path $\lambda : [0, 1] \to G$ with a nonzero vector v and then normalizing. From the formula for the section s we get the ith portion of the path, which we denote λ_i, as a concatenation of paths:

$$\lambda_i = A_i^{-1} B_i^{-1} A_i \beta * A_i^{-1} B_i^{-1} \alpha * A_i^{-1} \beta_i^{-1} * \alpha_i^{-1}$$

By Lemma 10, λ_i is homotopic to path $\alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = [\alpha_i^{-1}, \beta_i^{-1}]$. Gluing together the λ_i, we get that λ is homotopic to the path γ. Finally, Lemma 11 guarantees that the index of s is $\deg(\widehat{\gamma}) = \delta(\rho)$. □

Lemma 12 *The function $\widetilde{\theta} : \widetilde{G} \to \mathbb{R}$ satisfies the property:*

$$|\widetilde{\theta}(\alpha\beta) - \widetilde{\theta}(\alpha) - \widetilde{\theta}(\beta)| < \pi/2.$$

Proof Given $A, B \in G$ there are symmetric positive definite matrices R, T such that:

$$AB = \theta(A) R \theta(B) T.$$

If we set $X = \theta(B)^{-1} R \theta(B)$ then:

$$AB = \theta(A)\theta(B) X T.$$

This implies that:

$$\theta(AB) = \theta(A)\theta(B)\theta(XT).$$

Therefore:

$$\theta(B)^{-1}\theta(A)^{-1}\theta(AB) = \theta(XT)$$

Since X, T are positive definite, Exercise 1 implies that $\theta(B)^{-1}\theta(A)^{-1}\theta(AB)$ has positive trace. Fix $\alpha, \beta \in \widetilde{G}$ such that $p(\alpha) = A$ and $p(\beta) = B$ and set $\Delta = \widetilde{\theta}(\alpha\beta) - \widetilde{\theta}(\alpha) - \widetilde{\theta}(\beta)$. Then:

$$\phi(\Delta) = \begin{bmatrix} \cos\Delta & \sin\Delta \\ -\sin\Delta & \cos\Delta \end{bmatrix} = \theta(B)^{-1}\theta(A)^{-1}\theta(AB)$$

has positive trace. So $\cos\Delta > 0$. Since Δ is a continuous function of α and β which vanishes when $\alpha = \beta = 1$ we conclude that $|\Delta| < \pi/2$.

□

Lemma 13 *Given a representation $\rho : \pi_1(\Sigma) \to G$ we have:*

$$|\delta(\rho)| < g.$$

Proof Applying Lemma 12 $4g-1$ times one obtains:

$$|\widetilde{\theta}([\alpha_1,\beta_1]\dots[\alpha_g,\beta_g])-\widetilde{\theta}(\alpha_1)-\widetilde{\theta}(\beta_1)-\widetilde{\theta}(\alpha_1^{-1})-\widetilde{\theta}(\beta_1^{-1})\cdots \\ -\widetilde{\theta}(\alpha_g)-\widetilde{\theta}(\beta_g)-\widetilde{\theta}(\alpha_g^{-1})-\widetilde{\theta}(\beta_g^{-1})|<(4g-1)\frac{\pi}{2}$$

By Exercise 1 we know that

$$\widetilde{\theta}(\alpha)+\widetilde{\theta}(\alpha^{-1})=0,$$

so that the inequality above becomes:

$$|\widetilde{\theta}([\alpha_1,\beta_1]\dots[\alpha_g,\beta_g])|<(4g-1)\frac{\pi}{2}.$$

Dividing by 2π one obtains:

$$|\delta(\rho)|<g-\frac{1}{4}<g.$$

□

The lemma gives one direction of Milnor's inequalities. We will now show that the converse is also true. Let us set $A_0:=\begin{pmatrix}2&0\\0&1/2\end{pmatrix}$ and $\alpha_0\in p^{-1}(A_0)$ so that $\widetilde{\theta}(\alpha_0)=0$. We will denote by K and $\widetilde{K}$ the conjugacy classes of A_0 and α_0 respectively.

Since $\mathbb{R}$ is simply connected, the group homomorphism $\phi:\mathbb{R}\to S\subset G$ lifts to a homomorphism $\mathbb{R}\to\widetilde{G}$ and therefore $\mathbb{R}$ acts on $\widetilde{G}$ and also on conjugacy classes of $\widetilde{G}$.

Lemma 14 *Any element of $\pi\widetilde{K}$ can be written as a product of two elements in $\widetilde{K}$.*

Proof Consider the matrices:

$$A_1=\begin{pmatrix}-5/2&9/2\\-3&5\end{pmatrix}$$

and

$$A_2=A_0A_1=\begin{pmatrix}-5&9\\-3/2&5/2\end{pmatrix}.$$

Since the trace and determinant characterise the sets K and πK, then A_0, $A_1\in K$ and $A_2\in\pi K$. Let $\alpha_0,\alpha_1\in\widetilde{K}$ be such that $p(\alpha_i)=A_i$ and $\widetilde{\theta}(\alpha_i)=0$. Then $\alpha_0\alpha_1\in n\pi\widetilde{K}$ for some odd integer n. Let us show that $n=\pm1$. Given $\alpha\in\widetilde{K}$ we know that $p(\alpha)$

has positive trace and therefore $\cos(\widetilde{\theta}(\alpha)) > 0$. Therefore, if $\beta \in n\pi\widetilde{K}$ with $|n| > 1$ then:

$$|\widetilde{\theta}(\beta)| > \frac{5\pi}{2}.$$

Applying Lemma 12 we obtain:

$$|\widetilde{\theta}(\alpha_0\alpha_1)| < |\widetilde{\theta}(\alpha_0)| + |\widetilde{\theta}(\alpha_1)| + \frac{\pi}{2} < \frac{3}{2}\pi$$

We conclude that $n = \pm 1$. Now:

- If $n = 1$ we set:

$$\gamma(\alpha_0\alpha_1)\gamma^{-1} = (\gamma\alpha_0\gamma^{-1})(\gamma\alpha_1\gamma^{-1}) \in \widetilde{K}\widetilde{K}.$$

- If $n = -1$ then $\gamma(\alpha_0\alpha_1)\gamma^{-1} \in -\pi\widetilde{K}$ and $\gamma(\alpha_0\alpha_1)^{-1}\gamma^{-1} \in \pi\widetilde{K}$. We then write:

$$\gamma(\alpha_0\alpha_1)^{-1}\gamma^{-1} = (\gamma\alpha_1^{-1}\gamma^{-1})(\gamma\alpha_0^{-1}\gamma^{-1}) \in \widetilde{K}\widetilde{K}.$$

□

Lemma 15 *For every $\alpha \in \pi\widetilde{K}$ there are elements $\beta_1\beta_2 \in \widetilde{G}$ such that $\alpha = \beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$.*

Proof By the previous lemma we have $\beta_1, \beta_3 \in \widetilde{K}$ such that $\alpha = \beta_1\beta_3$. Since $\beta_1^{-1} \in \widetilde{K}$, there exists $\beta_2 \in \widetilde{G}$ so that $\beta_2\beta_1^{-1}\beta_2^{-1} = \beta_3$.

Then

$$\alpha = \beta_1\beta_3 = \beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$$

□

Lemma 16 *For every $n \geq 1$ there are $\gamma_1, \ldots, \gamma_{n+1} \in \widetilde{K}$ so that $\gamma_1 \ldots \gamma_{n+1} = n\pi\alpha_0$.*

Proof We will argue by induction. The case $n = 1$ is Lemma 14. Now assume that the lemma holds for n so that we can write:

$$\gamma_1 \ldots \gamma_{n+1} = n\pi\alpha_0.$$

Then:

$$(n+1)\pi\alpha_0 = \pi(n\pi)\alpha_0 = \pi(\gamma_1 \ldots \gamma_{n+1}) = (\pi\gamma_1)\ldots\gamma_{n+1} = \gamma\gamma'\gamma_2\ldots\gamma_{n+1}.$$

Here $\gamma, \gamma' \in \widetilde{K}$ and we have applied the Lemma 14 to $\pi\gamma_1$. □

Theorem 5 *A rank two oriented vector bundle $\pi : E \to \Sigma$ admits a flat connection if and only if*

$$|D(E)| < g.$$

Proof One direction is guaranteed by Lemma 13. We want to prove that a bundle E with $|D(E)| < g$ admits a flat connection. It suffices to show that there is some representation $\rho : \pi_1(\Sigma) \to G$ such that $\delta(\rho) = D(E)$. Since the existence of a flat connection does not depend on the orientation we may assume that $D(E) > 0$. Furthermore, some of the matrices in the representation can be set to equal the identity, therefore it suffices to show the case where $D(E) = g - 1$ and $g > 1$. By Lemma 16 there exist $\gamma_1, ..., \gamma_{g-1} \in \widetilde{K}$ so that:

$$\gamma_1 \dots \gamma_{g-1} = (g-2)\pi\alpha_0.$$

If we set $\gamma_g = \alpha_0^{-1}$ then:

$$\gamma_1 \dots \gamma_{g-1}\gamma_g = (g-2)\pi.$$

Now by Lemma 15, for every $1 \leqslant i \leqslant g$ there exists $\alpha_i, \beta_i \in \widetilde{G}$ so that

$$[\alpha_i, \beta_i] = \pi\gamma_i.$$

Then

$$\prod_{i=1}^{g}[\alpha_i, \beta_i] = 2\pi(g-1).$$

This implies that by setting $A_i = p(\alpha_i)$ and $B_i = p(\beta_i)$ one obtains a representation ρ such that:

$$\delta(\rho) = \frac{1}{2\pi}\prod_{i=1}^{g}[\alpha_i, \beta_i] = g - 1.$$

□

Corollary 3 *The torus is the only closed oriented surface that admits an affine structure. In particular, Chern's conjecture holds in dimension $d = 2$.*

Proof Since S^2 is simply connected, a flat connection on TM would give a trivialization of TM, which does not exist because the Euler characteristic of the sphere is 2. Let us now assume that $g > 1$. Then $D(TM) = \chi(\Sigma) = 2 - 2g$. Therefore:

$$|D(TM)| = 2|(1-g)| = 2g - 2 \geq g.$$

□

2.2 The Counterexamples of Smillie

Chern's conjecture asks about the Euler characteristic of closed affine manifolds. For a while, it was not known whether the condition that the connection is torsion free was essential. A manifold M is said to be flat if its tangent bundle admits a flat (not necessarily torsion free) connection. A strong version of Chern's conjecture asking whether the Euler characteristic of a closed flat manifold vanishes was an open problem for a while. This question was answered by Smillie [17], who proved the following

Theorem (Smillie)
For each $n > 1$ there are closed flat manifolds M^{2n} which have non-zero Euler characteristic.

In what follows we will present Smillie's construction.

Definition 9 Abusing the notation, the trivial vector bundle of rank n over M will be denoted $\mathbb{R}^n$. It will be clear from the context whether we are referring to the trivial bundle or the euclidean space. We will say that a vector bundle $\pi : E \to M$ is almost trivial if

$$E \oplus \mathbb{R} \simeq \mathbb{R}^m.$$

We will say that M is almost parallelizable if TM is almost trivial.

Lemma 17 *An oriented vector bundle $\pi : E \to M$ is almost parallelizable if and only if there exists a function $f : M \to S^m$ such that:*

$$f^*(TS^m) \simeq E.$$

Proof Since TS^m is almost parallelizable, so is $f^*(TS^m)$. On the other hand, suppose that E is almost parallelizable. Once we fix an isomorphism

$$\varphi : E \oplus \mathbb{R} \longrightarrow \mathbb{R}^{m+1},$$

there is a unique smooth function $f : M \to S^m$ such that:

- $f(p) \in (\varphi(E_p))^{\perp}$.
- If $\{e_1, \ldots, e_m\}$ is an oriented basis for E_p then

$$\{f(p), \varphi(e_1), \ldots, \varphi(e_m)\}$$

is an oriented basis for $\mathbb{R}^{m+1}$.

By construction φ restricts to an isomorphism from E to $f^*(TS^m)$. □

Exercise 4 Show that there exists an embedding:

$$\iota : S^m \times S^n \to \mathbb{R}^{m+n+1}$$

with trivial normal bundle. Conclude that $S^m \times S^n$ is almost parallelizable.

Lemma 18 *Let $\pi_1 : E \to M$ and $\pi_2 : E' \to N$ be almost trivial vector bundles. Then the vector bundle $p_1{}^*E \oplus p_2{}^*E'$ over $M \times N$ is almost trivial.*

Proof By Lemma 17, there are functions $f : M \to S^m$ and $g : N \to S^n$ such that

$$E \simeq f^*(TS^m); \text{ and } E' \simeq g^*(TS^n).$$

By Exercise 4, there is a function $h : S^m \times S^n \to S^{m+n}$ such that

$$h^*(TS^{m+n}) = T(S^m \times S^n).$$

Then we have

$$(h \circ (f \times g))^*(TS^{m+n}) = (f \times g)^*(h^*(TS^{m+n})) = (f \times g)^*(T(S^m \times S^n)) = p_1{}^*E \oplus p_2{}^*E.$$

By Lemma 17, we conclude that $p_1{}^*E \oplus p_2{}^*E$ is almost trivial. □

Definition 10 Let M, N be closed oriented manifolds of dimension m and $f : M \to N$ be a smooth function. The degree of f is the number:

$$\mathsf{deg}(f) := \int_M f^*(\omega)$$

where ω is any m-form on N such that:

$$\int_N \omega = 1.$$

Exercise 5 Show that the number $\mathsf{deg}(f)$ defined above is an integer that does not depend on the choice of ω. Show also that if f and g are smoothly homotopic then they have the same degree.

It turns out that the degree is a complete invariant of maps from a closed oriented m-manifold to S^m. This is a theorem of Heinz Hopf, whose proof can be found in [2].

Theorem 6 (Hopf Degree Theorem) *Let M be a closed, connected, oriented m-dimensional manifold and $f, g : M \to S^m$ be smooth maps. Then f and g are smoothly homotopic if and only if they have the same degree.*

Lemma 19 *Let M be a closed oriented manifold of dimension m and E, E' oriented almost trivial vector bundles of rank m. Then E and E' are isomorphic if and only if they have the same Euler class.*

Proof Clearly, if the bundles are isomorphic they have the same Euler class. Let us prove the converse. By Lemma 17, we know that there are functions $f, g : M \to S^m$ such that

$$f^*(TS^m) \simeq E; \text{ and } g^*(TS^m) \simeq E'.$$

Since E and E' have the same Euler class we conclude that f and g have the same degree. The Hopf degree theorem implies that f and g are smoothly homotopic and therefore E is isomorphic to E'. □

Lemma 20 *The connected sum of almost parallelizable manifolds is almost parallelizable.*

Proof Let $f : M \to S^m$ and $g : N \to S^m$ be smooth maps such that

$$f^*(TS^m) \simeq TM \text{ and } g^*(TS^m) \simeq TN.$$

Fix $p \in M$ and $q \in N$ and coordinates $\varphi : U \to \mathbb{R}^m$ and $\phi : W \to \mathbb{R}^m$ around p and q respectively. Set $X = \varphi^{-1}(B(0,1)) \subseteq U$ and $Y = \phi^{-1}(B(0,1)) \subseteq W$. Since $\pi_{m-1}(S^m) = 0$ we may assume that

$$(f \circ \varphi^{-1})|_{S^{m-1}} = (g \circ \phi^{-1})|_{S^{m-1}}.$$

Then there is a well defined map

$$h : M\#N \to S^m$$

such that $h|_{M-X} = f$ and $h|_{N-Y} = g$. We conclude that

$$h^*(TS^m) \simeq T(M\#N)$$

and therefore $M\#N$ is almost parallelizable. □

Theorem 7 (Smillie) *Let Σ_g be the closed oriented surface of genus g and $P = S^1 \times S^3$. Then*

$$M^4 = (\Sigma_3 \times \Sigma_3)\#\underbrace{P\#\cdots\#P}_{6 \text{ times}}$$

and

$$M^6 = ((\Sigma_3 \times \Sigma_3)\#\underbrace{P\#\cdots\#P}_{9 \text{ times}}) \times \Sigma_3$$

are closed flat manifolds with non-zero Euler characteristic. By taking products of M^4 and M^6 one obtains closed flat manifolds with nonzero Euler characteristic in all even dimensions greater than $d = 2$.

Proof The Euler characteristic of a connected sum of closed even dimensional manifolds satisfies:

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

Using this formula, we compute

$$\chi(M^4) = \chi(\Sigma_3 \times \Sigma_3) + \chi(\underbrace{P\#\cdots\#P}_{6 \text{ times}}) - 2 = 16 - 10 - 2 = 4.$$

and

$$\chi(M^6) = (\chi(\Sigma_3 \times \Sigma_3) + \chi(\underbrace{P\#\cdots\#P}_{9 \text{ times}}) - 2) \times \chi(\Sigma_3) = (16 - 16 - 2) \times (-4) = 8.$$

It remains to prove that M^4 and M^6 are flat manifolds. Lemmas 20 and 18 imply that M^4 and M^6 are almost parallelizable. Let $h : \Sigma_3 \to S^2$ be a degree one map and set $E = h^*(TS^2)$. Then E is almost parallelizable and

$$\int_{\Sigma_3} e(E) = \int_{S^2} e(TS^2) = 2.$$

Lemma 18 implies that $\pi^*E \oplus \pi^*E$ is an almost trivial bundle over $\Sigma_3 \times \Sigma_3$. Let

$$f : M^4 \to \Sigma_3 \times \Sigma_3$$

be the map that sends

$$\underbrace{P\#\cdots\#P}_{6 \text{ times}}$$

to a point. Then f has degree 1 and therefore:

$$\int_{M^4} e(f^*(E \oplus E)) = \int_{M^4} f^*(e(\pi(E \otimes E)) = \int_{\Sigma_3 \times \Sigma_3} \pi^*(e(E)) \wedge \pi^*(e(E)) = \Big(\int_{\Sigma_3} e(E)\Big)^2 = 4.$$

We conclude that:

$$e(TM^4) = e(\pi^*E \oplus \pi^*(E))$$

and thus, by Lemma 19

$$TM^4 \simeq {\pi_1}^*E \oplus {\pi_2}^*(E).$$

Milnor's inequality stated in Theorem 5 implies that E admits a flat connection and therefore, so does TM^4. Similarly, Let

$$g : \Sigma_3 \times \Sigma_3 \# \underbrace{P\#\cdots\#P}_{9 \text{ times}} \to \Sigma_3 \times \Sigma_3$$

be the map that sends

$$\underbrace{P\#\cdots\#P}_{9\text{ times}}$$

to a point. Then g has degree one and therefore the map

$$z := g \times \mathsf{id} : M^6 \to \Sigma_3 \times \Sigma_3 \times \Sigma_3$$

also has degree one. This implies that

$$\int_{M^6} e(z^*(\pi^*E \oplus \pi^*E \oplus \pi^*E)) = \Big(\int_{\Sigma_3} e(E)\Big)^3 = 8.$$

We conclude that

$$e(TM^6) = e(\pi^*E \oplus \pi^*E \oplus \pi^*E)$$

and therefore

$$TM^6 \simeq \pi^*E \oplus \pi^*E \oplus \pi^*E.$$

Theorem 5 implies that E admits a flat connection and therefore, so does TM^6. □

2.3 The Case of Complete Manifolds, After Sullivan and Kostant

Here we will prove the following theorem of Kostant and Sullivan which solves Chern's conjecture in the case of complete affine manifolds.

Theorem (Kostant–Sullivan)
If M is a closed affine complete manifold then $\chi(M) = 0$.

The idea of their proof is that even though the connection on M may not admit a compatible metric, the Chern–Weil theory of characteristic classes can still be used to prove that the Euler class vanishes.

Definition 11 We will say that a subgroup G of $\mathrm{GL}(m, \mathbb{R})$ is 1-spectral if for every $A \in G$,

$$\mathsf{det}(A - \mathsf{id}) = 0.$$

Definition 12 We will say that a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(m, \mathbb{R})$ is singular if for all $v \in \mathfrak{g}$,

$$\mathsf{det}(\mathsf{v}) = 0.$$

The key observation of Sullivan and Kostant is that if the frame bundle of TM admits a reduction to a connected compact and 1-spectral subgroup of $\mathrm{GL}^+(m,\mathbb{R})$ then the Euler class of TM vanishes.

Lemma 21 *If G is a 1-spectral Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$ then its Lie algebra $\mathfrak{g}$ is singular.*

Proof It is enough to prove that there is an open subset $W \subseteq \mathfrak{g}$ such that $0 \in W$ and all matrices of W are singular. Choose W such that the exponential map $\mathsf{Exp} : W \to U \subseteq G$ is a diffeomorphism with inverse $\mathsf{Log} : U \to W$. For $v \in W$, $A \in U$ these functions are given explicitly by

$$\mathsf{Exp}(v) = \sum_{k \geqslant 0} \frac{x^k}{k!},$$

$$\mathsf{Log}(A) = \sum_{k \geqslant 1} \frac{(-1)^{k+1}(A - \mathsf{id})^k}{k}.$$

Any $v \in W$ is of the form $v = \mathsf{Log}(A)$ with $\mathsf{det}(A - \mathsf{id}) = 0$. Thus there exists a vector $x \in \mathbb{R}^m$ such that $Ax = x$. Then

$$v(x) := \sum_{k \geqslant 1} \frac{(-1)^{k+1}(A - \mathsf{id})^k}{k}(x) = \sum_{k \geqslant 1} \frac{(-1)^{k+1}(A - \mathsf{id})^k(x)}{k} = 0.$$

We conclude that v is singular. □

Lemma 22 *Let M be a manifold of dimension m and P a $\mathrm{GL}^+(m,\mathbb{R})$ principal bundle over M that admits a reduction to a subgroup G which is compact, connected and 1-spectral. Then the Euler class of P vanishes.*

Proof The Chern–Weil homomorphism

$$\mu_P : H(B\mathrm{GL}^+(m,\mathbb{R})) \to H(M)$$

factors through the restriction map

$$\iota^* : H(B\mathrm{GL}^+(m,\mathbb{R})) \to H(BG).$$

Moreover, there is a commutative diagram

$$\begin{array}{ccc} H(B\mathrm{GL}^+(m,\mathbb{R})) & \xrightarrow{\iota^*} & H(BG) \\ \downarrow \simeq & & \downarrow \simeq \\ S(\mathfrak{gl}(m,\mathbb{R})^*)^{\mathrm{GL}^+(m,\mathbb{R})} & \xrightarrow{\rho} & S(\mathfrak{g}^*)^G \end{array}$$

where the vertical arrows are isomorphisms and ρ is the restriction of polynomials. Since the Euler class corresponds to the Pfaffian polynomial, it suffices to prove that $\rho(\mathsf{Pf}) = 0$. For this we observe that since $\mathfrak{g}$ is singular

$$[\rho(\mathsf{Pf})(v)]^2 = \det(v)^2 = 0.$$

We conclude that $\rho(\mathsf{Pf}) = 0$ and therefore the Euler class of P vanishes. □

Lemma 23 *Let M be a manifold of dimension m and P a $\mathrm{GL}^+(m, \mathbb{R})$ principal bundle over M that admits a reduction to a subgroup G which is closed, connected and 1-spectral. Then the Euler class of P vanishes.*

Proof Since G is a closed subgroup of the general lineal group, it is a Lie subgroup. Let $K \subseteq G$ be a maximal compact subgroup. Since G/K is contractible, the natural map $\pi : BK \to BG$ is a homotopy equivalence. The Chern–Weil homomorphism

$$\mu_P : H(B\mathrm{GL}^+(m, \mathbb{R})) \to M$$

factors through the restriction map

$$\iota^* : H(B\mathrm{GL}^+(m, \mathbb{R})) \to H(BG)$$

so it suffices to show that ι^* sends the Euler class to zero. There is also a commutative diagram

$$\begin{array}{ccc} H(B\mathrm{GL}^+(m, \mathbb{R})) & \xrightarrow{\pi^* \circ \iota^*} & H(BK) \\ \downarrow \simeq & & \downarrow \simeq \\ S(\mathfrak{gl}(m, \mathbb{R})^*)^{\mathrm{GL}^+(m,\mathbb{R})} & \xrightarrow{\rho} & S(\mathfrak{K}^*)^K \end{array}$$

By the Lemma 22, we know that $\rho(\mathsf{Pf}) = 0$ and therefore $\pi^* \circ \iota^*$ sends the Euler class to zero. Since π^* is an isomorphism, we conclude that ι^* sends the Euler class to zero. □

Lemma 24 *Let M be a manifold of dimension m and P a $\mathrm{GL}^+(m, \mathbb{R})$ principal bundle over M that admits a reduction to a subgroup $G \subseteq \mathrm{GL}^+(m, \mathbb{R})$ which is 1-spectral and closed. If G has finite number of connected components then the Euler class of P is zero.*

Proof As in the previous lemmas, it suffices to show that the restriction map:

$$\iota^* : H(B\mathrm{GL}^+(m, \mathbb{R})) \to H(BG)$$

sends the Euler class to zero. Let G_0 be the connected component of the identity in G. Then the natural map

$$\pi : BG_0 \to BG$$

is a finite covering and therefore it induces an injective map in cohomology. Therefore it suffices to show that the map

$$\pi^* \circ \iota^* : H(B\mathrm{GL}^+(m, \mathbb{R})) \to H(BG_0)$$

sends the Euler class to zero. This is guaranteed by Lemma 23.

□

In view of the lemma above, we are left with the problem of showing that the frame bundle of a closed complete affine manifold admits a reduction to a closed 1-spectral group with finitely many connected components.

Lemma 25 *Let G be a 1 spectral subgroup of $\mathrm{GL}(m, \mathbb{R})$ and $\overline{G}$ its Zariski closure. Then:*

1. *$\overline{G}$ is a subgroup of $\mathrm{GL}(m, \mathbb{R})$.*
2. *$\overline{G}$ is a closed Lie group in $\mathrm{GL}(m, \mathbb{R})$.*
3. *$\overline{G}$ is 1-spectral.*
4. *$\overline{G}$ has finitely many connected components.*

Proof Since the multiplication and inverse functions are algebraic operations, they are continuous in the Zariski topology. Fix $x \in G$ and consider the homeomorphism

$$\begin{array}{rccc} L_x : & \mathrm{GL}(m, \mathbb{R}) & \longrightarrow & \mathrm{GL}(m, \mathbb{R}) \\ & y & \longmapsto & xy \end{array}.$$

Then

$$x\overline{G} = \overline{xG} \subseteq \overline{G}.$$

Fix now $y \in \overline{G}$ and consider the homeomorphism R_y given by right multiplication by y. Then

$$\overline{G}y = \overline{Gy} \subseteq \overline{G}.$$

We conclude that $\overline{G}$ is closed with respect to the product. The map $x \mapsto x^{-1}$ is a homeomorphism of $\mathrm{GL}(m, \mathbb{R})$ and therefore:

$$(\overline{G})^{-1} = \overline{G^{-1}} = \overline{G}.$$

We conclude that G is a group. The second statement is true because $\overline{G}$ is closed in the Zariski topology and therefore also in the smooth topology. The third statement holds because the determinant is a continuous function. The last statement is true because any real algebraic set has finitely many connected components. □

We can now prove Chern's conjecture for complete manifolds:

Theorem 8 (Kostant–Sullivan) *If M is a closed affine complete manifold then $\chi(M) = 0$.*

Proof In view of Lemma 24, it suffices to show that the frame bundle of TM admits a reduction to a closed 1-spectral subgroup of $\mathrm{GL}^+(m, \mathbb{R})$ which has finitely many connected components. By Proposition 1, we know that M is the quotient $\mathbb{R}^m/\Gamma$ where $\Gamma \subset \mathsf{Aff}(\mathbb{R}^m)$ is isomorphic to the fundamental group of M. Consider the natural homomorphism

$$\lambda : \mathsf{Aff}(\mathbb{R}^m) \to \mathrm{GL}(m, \mathbb{R}),\ Ax + b \mapsto A$$

and let G be the group $\lambda(\Gamma)$. We claim that G is 1-spectral. Take $g = Ax + b \in \mathsf{Aff}(\mathbb{R}^m)$. If A is not 1-spectral then the equation $Ax + b = x$ has a solution, which is impossible because Γ acts freely on $\mathbb{R}^m$. Lemma 25 guarantees that $\overline{G}$ is a closed 1-spectral group with finitely many connected components. Therefore, it suffices to prove that the frame bundle of TM admits a reduction to the group G and therefore to $\overline{G}$. Consider the projection

$$\pi : \mathbb{R}^m \to M$$

and for each $p \in M$, the following subset of the frame bundle of T_pM

$$S_p := \{\phi : \mathbb{R}^m \to T_pM : \phi \text{ is a linear isomorphism of the form } \phi = D\pi(x) \text{ for } x \in \pi^{-1}(p)\}.$$

The group G acts on the right by composition and this action is free and transitive. If we set:

$$S := \coprod_p S_p \subset \mathsf{Fr}(TM)$$

we obtain a reduction of the structure group of the frame bundle of TM to G. □

3 The Case of Special Affine Manifolds, After Klingler

In the case of special affine manifolds, Chern's conjecture holds.

Definition 13 An affine manifold M is called special if it admits a covariantly constant volume form $\omega \in \Omega^m(M)$.

Lemma 26 *An affine manifold is special if and only if there exists an affine atlas for which the transition functions are of the form:*

$$f(x) = Ax + b$$

where $\det(A) = 1$. *We will call such an atlas a special atlas.*

Proof Suppose $\omega \in \Omega^m(M)$ is a parallel volume form. Then the family of affine coordinates $\varphi : U \to V \subseteq \mathbb{R}^m$ such that:

$$dx_1 \wedge \cdots \wedge dx_m = \omega|_U$$

is a special atlas. Conversely, given a special atlas $\{U_\alpha, \varphi_\alpha\}$ there is unique parallel volume form $\omega \in \Omega^m(M)$ such that for special affine coordinates:

$$dx_1 \wedge \cdots \wedge dx_m = \omega|_{U_\alpha}$$

□

In what follows we will present Klingler's proof [16] of the following remarkable theorem:

Theorem (Klingler)
The Euler characteristic of a closed special affine manifold is zero.

The structure of Klingler's proof is as follows. Proposition 2 shows that if M is an affine manifold then TM has a canonical para-hypercomplex structure and therefore there are spectral sequences $\mathcal{B}_r^{ij}$ and $\mathcal{B}_{c,r}^{ij}$ converging to $H(TM)$ and $H_c(TM)$ respectively. Proposition 3 shows that if the natural map induced by the inclusion:

$$\lambda : \mathcal{B}_{c,\infty}^{0m} \to \mathcal{B}_\infty^{0m}$$

is zero then the Euler characteristic of M is zero. Finally, Proposition 4 shows that if M is special affine then:

$$\lambda : \mathcal{B}_{c,\infty}^{0m} \to \mathcal{B}_\infty^{0m}$$

is the zero map.

3.1 Local Product Structures and Affine Manifolds

Definition 14 A local product structure on M is a pair of foliations $H, V \subset TM$ such that

$$TM = H \oplus V.$$

Vector fields tangent to H will be called horizontal and those tangent to V will be called vertical.

Of course, if $M = X \times Y$ then M has a local product structure given by:

$$H = \mathsf{Ker}(D\pi_2) \quad \text{and} \quad V = \mathsf{Ker}(D\pi_1).$$

Thus, a product structure induces a local product structure.

Let M be a manifold with a local product structure $H \oplus V = TM$. Then there is a natural decomposition

$$\Omega(M) = \Gamma(\Lambda(H \oplus V)^*) = \Gamma\left(\Lambda H^* \otimes \Lambda V^*\right).$$

We set

$$\Omega^{ij}(M) := \Gamma\left(\Lambda^i H^* \otimes \Lambda^j V^*\right),$$

so that

$$\Omega^k(M) = \bigoplus_{i+j=k} \Omega^{ij}(M).$$

A form $\eta \in \Omega^{ij}(M)$ vanishes when evaluated on more than i horizontal vector fields or more than j vertical vector fields.

Lemma 27 *Let M be a manifold with a local product structure $TM = H \oplus V$. Then the de-Rham operator*

$$d : \Omega(M) \to \Omega(M)$$

decomposes as a sum

$$d = d_H + d_V$$

where

$$d_H : \Omega^{ij}(M) \to \Omega^{i+1,j} \quad \text{and} \quad d_V : \Omega^{ij} \to \Omega^{i,j+1}(M).$$

Thus, $\Omega^{ij}(M)$ becomes a double complex.

Proof Recall that for a differential form $\eta \in \Omega^k(M)$ the de-Rham operator is given by the formula

$$\begin{aligned} d\eta(X_0, \cdots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\eta(X_0, \ldots, \hat{X}_i, \cdots, X_k)) \\ &+ \sum_{r<s} (-1)^{r+s} \eta([X_r, X_s], X_0, \ldots, \hat{X}_r, \ldots, \hat{X}_s, \ldots, X_k). \end{aligned}$$

Consider $\eta \in \Omega^{ij}(M)$ with $k = i + j$. We claim that

$$d\eta(X_0, \ldots, X_k) = 0$$

if there are more than $i + 1$ horizontal vector fields. Suppose this is the case. Then the list $X_0, \ldots, \hat{X}_i, \ldots, X_k$ contains at least $i + 1$ horizontal vector fields and therefore

$$\eta(X_0, \cdots, \hat{X}_i, \cdots, X_k) = 0.$$

Also, since H is a foliation, the list

$$[X_r, X_s], X_0, \cdots, \hat{X}_r, \ldots, \hat{X}_s, \ldots, X_k$$

contains at least $i + 1$ horizontal vector fields and therefore

$$\eta([X_r, X_s], X_0, \ldots, \hat{X}_r, \ldots, \hat{X}_s, \cdots, X_k) = 0.$$

By symmetry, it is also true that $d\eta(X_0, \ldots, X_k) = 0$ of there are more than $j + 1$ vertical vector fields. Thus, we conclude that

$$d\eta \in \Omega^{i+1,j}(M) \oplus \Omega^{i,j+1}(M),$$

as claimed. □

We have seen that the de-Rham complex of a manifold with a local product structure decomposes as a double complex. This induces two decreasing filtrations which we will denote as follows:

$$F_r^V(\Omega(M)) = \bigoplus_{j \geq r} \Omega^{ij}(M);$$

$$F_r^H(\Omega(M)) = \bigoplus_{i \geq r} \Omega^{ij}(M).$$

Notice that forms in $F_r^V(\Omega^k(M))$ are precisely the k-forms that vanish when evaluated in $k + 1 - r$ horizontal vector fields, hence this filtration is just the filtration associated to the horizontal foliation H. We will be concerned mostly with a special kind of local product structures called paracomplex structures, which we now introduce.

Definition 15 For any endomorphism of the tangent bundle $A \in \mathsf{End}(TM)$ the Nijenhuis tensor of A is defined by

$$N_A(X, Y) = -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY].$$

Exercise 6 Show that the Nijenhuis tensor is indeed a tensor.

Definition 16 An almost-complex structure on M is an endomorphism $I \in \mathsf{End}(TM)$ such that $I^2 = -\mathrm{id}$. A complex structure on M is an almost complex structure I such that $N_I = 0$. A para-complex structure on M is an endomorphism $J \in \mathsf{End}(TM)$ such that $J^2 = \mathrm{id}$ and such that $TM = H \oplus V$ where H and V are the eigenspaces of 1 and -1 respectively, and are integrable distributions of the same dimension. A para-hypercomplex structure on M is the data of a complex structure I and a para-complex structure J satisfying $IJ = -JI$.

An almost complex structure on M gives TM the structure of a complex vector bundle by:

$$zX = aX + bI(X)$$

where $z = a + bi \in \mathbb{C}$.

Lemma 28 *Let M be a manifold with a para-hypercomplex structure. Then for any $z = a + bi \in S^1 \subseteq \mathbb{C}$, the endomorphism $J_z \in \mathsf{End}(TM)$ given by*

$$J_z(X) := zJ(X).$$

is a para-complex structure.

Proof First we observe that

$$J_z^2 = zJ(zJ) = z\bar{z}J^2 = \mathrm{id}.$$

Let H and V denote the eigenspaces of J associated to 1 and -1 respectively. Then the eigenspaces of J_z are $(1+z)H$ and $(1+z)V$. Let us check that these distributions are involutive. Take $X, Y \in H$ and write $z = a + bi$. Then

$$\begin{aligned}[(1+z)X, (1+z)Y] =& (1+a)[X,Y] + b\big([IX,Y] + [X,IY]\big) + \\ & a\big((1+a)[X,Y] + b\big([IX,Y] + [X,IY]\big)\big) + \\ & b^2[IX,IY]\end{aligned}$$

Since we have $N_I = 0$, we may replace $[IX,Y] + [X,IY] = i[X,Y] - I[IX,IY]$ in the previous expression, which yields

$$\begin{aligned}[(1+z)X, (1+z)Y] =& (1+a+bI)[X,Y] + a(1+a+bI)[X,Y] + \\ & - bI[IX,IY] - abI[IX,IY] + b^2[IX,IY]\end{aligned}$$

Ultimately, we get

$$[(1+z)X, (1+z)Y] = (1+z)[X,Y] + a(1+z)[X,Y] + -b(1+z)I[IX,IY].$$

The anticommutativity relation $JI = -IJ$ implies that I interchanges the eigenspaces H and V. In particular we have that $I[IX, IY] \in H$. Therefore $[(1+z)X, (1+z)Y] = (1+z)Z$ with $Z \in H$. A similar argument shows that $(1+z)V$ is also involutive. □

Proposition 2 *Let M be an affine manifold. Then TM has a canonical para-hypercomplex structure.*

Proof Fix affine coordinates $x_1, \ldots, x_m$ in M which determine coordinates $x_1, \ldots, x_m, y_1, \ldots, y_m$ in TM and set

$$J\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial x_i}, \quad J\Big(\frac{\partial}{\partial y_i}\Big) = -\frac{\partial}{\partial y_i},$$

and

$$I\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial y_i}, \quad I\Big(\frac{\partial}{\partial y_i}\Big) = -\frac{\partial}{\partial x_i}.$$

It is clear that $IJ + JI = 0$, $I^2 = -\mathrm{id}$ and the eigenspaces of J are involutive. Evaluating N_I in any couple of generators leads to a sum of terms of the form $\Big[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big]$, $\Big[\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\Big]$ or $\Big[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\Big]$, all of which vanish. Finally, since the transition functions are affine, the operators I and J are well defined. □

3.2 The Spectral Sequence and the Euler Class

As described in the previous section, an affine structure on a manifold M induces a para-hypercomplex structure on its tangent bundle TM. Klingler's proof relies heavily on the spectral sequences associated to this para-hypercomplex structure.

Given a manifold M we will denote by:

$$\lambda : \Omega_c(M) \to \Omega(M)$$

the inclusion of differential forms with compact support into all forms. The vanishing of the Euler characteristic can be reformulated as follows.

Lemma 29 *Let M be a connected closed manifold of dimension m and E an oriented vector bundle over M of rank r. The Euler class $e(E)$ is zero if and only if the natural morphism*

$$\lambda : H^r_c(E, \mathbb{R}) \to H^r(E, \mathbb{R})$$

vanishes. Here $H_c(E, \mathbb{R})$ denotes the cohomology with compact support of E.

Proof Let $\tau \in H_c(E, \mathbb{R})$ be the Thom class of E. By the Thom isomorphism

$$H_c^r(E, \mathbb{R}) \simeq H^0(M, \mathbb{R}) \simeq \mathbb{R}$$

and τ generates $H_c(E, \mathbb{R})$. If $\sigma \in \Gamma(E)$ is the zero section then

$$e(E) = \sigma^*(\lambda(\tau)).$$

Since

$$\sigma^* : H(E) \to H(M)$$

is an isomorphism, we conclude that the map λ is zero if and only if $e(E) = 0$. □

Let M be a closed affine manifold. We know that TM is a para-hypercomplex manifold.

- We will denote by $\mathcal{B}_r^{pq}$ the spectral sequence associated with the filtration $F^V(\Omega(TM))$.
- We will denote by $\mathcal{B}_{c,r}^{pq}$ the spectral sequence associated with the filtration $F^V(\Omega_c(TM))$.

Proposition 3 *Let M be a closed connected affine manifold of dimension m. If the morphism*

$$\lambda : \mathcal{B}_{c,\infty}^{0m} \to \mathcal{B}_\infty^{0m}$$

is zero then the Euler characteristic of M is zero.

Proof By the Thom isomorphism we know that $H_c^m(TM)$ is a one dimensional vector space and therefore

$$\bigoplus_{i+j=m} \mathcal{B}_{c,\infty}^{ij}$$

is also a one dimensional vector space. Let ψ be the composition of the following maps:

$$\bigoplus_{i+j=m} \mathcal{B}_{c,\infty}^{ij} \xrightarrow{\lambda} \bigoplus_{i+j=m} \mathcal{B}_\infty^{ij} \xrightarrow{c} \mathsf{Gr}(H^m(TM)) \xrightarrow{\tau} H^m(TM) \xrightarrow{\sigma^*} H^m(M).$$

The map c is the isomorphism given by the convergence of the spectral sequence. The map τ is the identification between $H^m(M)$ and its associated graded, which exists because $H^m(TM)$ is one dimensional. The map σ^* is the pull-back by the zero section. Since

$$\mathcal{B}_{c,0}^{ij} = \Omega_c^{ji}(TM),$$

an element in $\mathcal{B}^{ij}_{c,\infty}$ is represented by a class $[\omega_{ij}]$ where $\omega_{ij} \in \Omega^{ji}_c(TM)$. Hence, the map ψ is given by

$$\psi[\omega_{ij}] = \sigma^*(\omega_{ij}).$$

This implies that

$$\psi : \mathcal{B}^{ij}_{c,\infty} \to H^m(M)$$

is the zero map if $i > 0$. Since the maps c, τ and σ^* are isomorphisms we obtain that the map

$$\lambda : \mathcal{B}^{ij}_{c,\infty} \to \mathcal{B}^{ij}_{\infty}$$

is zero for $i > 0$. Furthermore, the following diagram is commutative

$$\begin{array}{ccc}
\bigoplus_{i+j=m} \mathcal{B}^{ij}_{c,\infty} & \xrightarrow{\lambda} & \bigoplus_{i+j=m} \mathcal{B}^{ij}_{\infty} \\
\downarrow c & & \downarrow c \\
\mathsf{Gr}(H^m_c(TM)) & \xrightarrow{\lambda} & \mathsf{Gr}(H^m(TM)) \\
\downarrow \tau & & \downarrow \tau \\
H^m_c(TM) & \xrightarrow{\lambda} & H^m(TM)
\end{array}$$

therefore the bottom map is zero if an only if the top map is zero if an only if the map

$$\lambda : \mathcal{B}^{0m}_{c,\infty} \to \mathcal{B}^{0m}_{\infty}$$

is zero. Lemma 29 then implies the claimed result. □

3.3 A Lemma About Invariant Forms

Let V be a finite dimensional real vector space. The vector space

$$TV = V \oplus V$$

has a natural complex structure given by

$$I(v, w) := (-w, v)$$

and therefore it is naturally a complex vector space. The group $\mathsf{Aff}(V)$ acts naturally on TV via the derivative by the formula

$$g \cdot (v, w) = (g(v), A(w))$$

for $g \in \mathsf{Aff}(V)$ of the form $g(v) = Av + b$. $S^1 \subseteq \mathbb{C}$ acts on TV by rotation

$$\theta(v, w) = (\cos\theta v - \sin\theta w, \sin\theta v + \cos\theta w), \quad e^{i\theta} \in S^1.$$

For each θ, we denote by TV_θ the vector space TV with the action of $\mathsf{Aff}(V)$ twisted by θ

$$g_\theta(v, w) = \theta^{-1} g(\theta(v, w)).$$

Lemma 30 *The vector space of invariant forms $\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$ is independent of θ and has dimension one.*

Proof Let us first prove that for $\theta = 1$ the space $\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$ is one dimensional. Fix a basis $v_1, \ldots, v_m$ for V which determines coordinates: $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ on TV. We will show that $\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$ is generated by $\omega = dy_1 \wedge \cdots \wedge dy_m$. Let us first prove that $\omega \in \Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$. We compute

$$\begin{aligned} g^*(\omega)(X_1, \ldots, X_m) &= \omega(AX_1, \ldots, AX_m) \\ &= \mathsf{det}(A)\omega(X_1, \ldots, X_m) = \omega(X_1, \ldots, X_m). \end{aligned}$$

We conclude that $g^*(\omega) = \omega$. It remains to prove that the set of invariant forms is one dimensional. Fix a point $(v, w) \in TV$ with $w \neq 0$ and an invariant form η. Then since $\Lambda^m(V)$ is a one dimensional vector space we know that at that point η is a multiple of ω. We may assume that they coincide at that point (v, w). Now the set W of points $(x, y) \in TV$ such that $y \neq 0$ open and dense, and moreover the group $\mathsf{SAff}(V)$ acts transitively on it. Since both η and ω are invariant it follows that they coincide on W. Since W is dense, they coincide on TV. We still need to prove that $\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$ is independent of θ.

Given $g(v) = Av + b \in \mathsf{SAff}(V)$ the derivative

$$Dg(p) : V \oplus V \to V \oplus V$$

is a complex linear map and therefore

$$\begin{aligned} g_\theta^*(\omega)(X_1, \ldots, X_m) &= \omega(AX_1, \ldots, AX_m) \\ &= \mathsf{det}(A)\omega(X_1, \ldots, X_m) = \omega(X_1, \ldots, X_m). \end{aligned}$$

This shows that ω belongs to $\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$. Since TV_θ is isomorphic to TV_1 we know that $\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$ is also one dimensional and therefore:

$$\Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)} = \Omega^{0m}(TV_1)^{\mathsf{SAff}(V)}.$$

□

3.4 Spectral Sequences of Sheaves

Let M be an affine manifold so that TM has a natural para-hypercomplex structure and for each $z \in S^1$ there is a decomposition of the de-Rham complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d_V^z} & & \uparrow{\scriptstyle d_V^z} & & \uparrow{\scriptstyle d_V^z} & & \\
\Omega^{02}(TM) & \xrightarrow{d_H^z} & \Omega^{12}(TM) & \xrightarrow{d_H^z} & \Omega^{22}(TM) & \xrightarrow{d_H^z} & \cdots \\
\uparrow{\scriptstyle d_V^z} & & \uparrow{\scriptstyle d_V^z} & & \uparrow{\scriptstyle d_V^z} & & \\
\Omega^{01}(TM) & \xrightarrow{d_H^z} & \Omega^{11}(TM) & \xrightarrow{d_H^z} & \Omega^{21}(TM) & \xrightarrow{d_H^z} & \cdots \\
\uparrow{\scriptstyle d_V^z} & & \uparrow{\scriptstyle d_V^z} & & \uparrow{\scriptstyle d_V^z} & & \\
\Omega^{00}(TM) & \xrightarrow{d_H^z} & \Omega^{10}(TM) & \xrightarrow{d_H^z} & \Omega^{20}(TM) & \xrightarrow{d_H^z} & \cdots
\end{array}$$

We will denote by $\mathcal{B}(z)_{c,r}^{pq}$ and $\mathcal{B}(z)_r^{pq}$ the spectral sequences associated to the vertical filtrations of these double complexes with compact and arbitrary support, respectively.

Definition 17 For each $z \in S^1$ we define the complex of sheaves

$$L_z^p := \mathsf{ker}(d_H^z : \Omega^{0p} \to \Omega^{1p}),$$

with differential d_V^z.

Exercise 7 Let M be a connected manifold. Show that there is a canonical difeomorphism between the tangent bundle of the universal cover of M and the universal cover of the tangent bundle of M

$$\widetilde{TM} \cong T(\widetilde{M}).$$

Definition 18 Let M be an affine manifold and $\varphi : \widetilde{M} \to V$ a developing map. The submersion $\psi : T(\widetilde{M}) \times S^1 \to TV = V \oplus V$ is defined by

$$\psi(\gamma, e^{i\theta}) := R_\theta \circ D\varphi(\gamma),$$

where R_θ is the rotation matrix

$$R_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathsf{GL}(V \oplus V).$$

Definition 19 We define the complex of sheaves $\mathcal{L}^p$ over $TM \times S^1$ as the descent of the $\pi_1(TM)$ equivariant complex of sheaves $\psi^{-1}(L_1^p)$ over $T(\widetilde{M}) \times S^1$. Also, we define the bicomplex of sheaves $\Omega^{pq}_{TM\times S^1}$ as the descent of the equivariant bicomplex $\psi^{-1}(\Omega^{pq}(TV), d_H + d_V)$.

Remark 1 The complex of sheaves L_1^p over TV resolves the constant sheaf $\mathbb{R}_{TV}$ and is quasi-isomorphic to the total complex of the double complex complex $\Omega^{pq}(TV)$. Since the functor ψ^{-1} is exact we conclude that the complex $\mathcal{L}^p$ resolves the constant sheaf and is quasi-isomorphic to the total complex of $\Omega^{pq}_{TM\times S^1}$. For each $z \in S^1$ the restriction of $\mathcal{L}^p$ to $TM \times \{z\}$ is the complex of sheaves L_z^p and the restriction of $\Omega^{pq}(TV)$ is the bicomplex $(\Omega^{pq}(TM), d_H^z + d_V^z)$.

Definition 20 Consider the second projection $p_2 : TM \times S^1 \to S^1$.

- We will denote by $\mathcal{E}^{pq}_{c,r}$ the spectral sequence computing

$$Rp_{2!}(\mathbb{R}) = Rp_{2!}(\mathcal{L}^\bullet).$$

- We will denote by $\mathcal{E}^{pq}_{r}$ the spectral sequence computing

$$Rp_{2*}(\mathbb{R}) = Rp_{2*}(\mathcal{L}^\bullet).$$

- We will denote by $\lambda : \mathcal{E}^{pq}_{c,r} \to \mathcal{E}^{pq}_{r}$ the morphism of spectral sequences of sheaves induced by the morphism of functors $Rp_{2!} \to Rp_{2*}$.

Lemma 31 *Let M be a closed affine manifold. Then the map*

$$\lambda : \mathcal{B}^{0m}_{c,\infty} \to \mathcal{B}^{0m}_{\infty}$$

factors through the vector space $(\mathcal{E}^{0m}_{\infty})_{z=1}$.

Proof By Remark 3, we know that for each $z \in S^1$ there are natural maps of spectral sequences:

$$\alpha : (\mathcal{E}^{pq}_{c,r})_z \to \mathcal{B}(z)^{pq}_{c,r}; \quad \beta : (\mathcal{E}^{pq}_{r})_z \to \mathcal{B}(z)^{pq}_{r}.$$

Moreover, α is an isomorphism and β is injective. Let us fix $z = 1$. Since these morphisms are natural we obtain that the map

$$\lambda : \mathcal{B}^{0m}_{c,\infty} \to \mathcal{B}^{0m}_{\infty}$$

can be factored as the following composition:

$$\mathcal{B}^{0m}_{c,\infty} \xrightarrow{\alpha^{-1}} (\mathcal{E}^{0m}_{c,\infty})_{z=1} \xrightarrow{\lambda} (\mathcal{E}^{0m}_{\infty})_{z=1} \xrightarrow{\beta} \mathcal{B}^{0m}_{\infty} .$$

□

Lemma 32 *The exists a unique open set $j : U \hookrightarrow S^1$ such that the sheaf $\mathcal{E}^{m0}_\infty$ is isomorphic to the sheaf $j_!(\mathbb{R}_U)$.*

Proof The sheaf $\mathcal{E}^{m0}_\infty$ is a constructible subsheaf of $\mathbb{R}_{S^1}$ and therefore its support U is open in S^1 and there is an isomorphism $\mathcal{E}^{m0}_\infty \simeq j_!(R_U)$. □

Proposition 4 *Let M be an closed special affine manifold. Then*

$$\mathcal{E}^{0m}_\infty = 0.$$

Proof It suffices to show that the inclusion morphism $\mathcal{E}^{m0}_\infty \hookrightarrow \mathbb{R}_{S^1}$ is an isomorphism. By Lemma 30 we can choose a nonzero element $\omega \in \Omega^{0m}(TV_\theta)^{\mathsf{SAff}(V)}$ which defines a global section $\overline{\omega}$ of the sheaf $\Omega^{0m}_{TM\times S^1}$. Since ω is closed, so is $\overline{\omega}$. This defines a global section of the sheaf $\mathcal{E}^{m0}_1$ and therefore also a section $[\omega]$ of its quotient $\mathcal{E}^{m0}_\infty$. Let us proof that $[\omega]$ is nonzero. For each $z \in S^1$ consider the composition

$$\mathcal{E}^{m0}_\infty(S_1) \longrightarrow (\mathcal{E}^{m0}_\infty)_z \xrightarrow{\beta} \mathcal{B}(z)^{m0}_\infty,$$

where the first morphism is given by taking the stalk of a section. This map sends $[\omega] \in \mathcal{E}^{m0}_\infty(S_1)$ to $[[\omega]] \in \mathcal{B}(z)^{m0}_\infty$. Let us set $z = \sqrt{-1}$. Then $\mathcal{B}(\sqrt{-1})^{m0}_\infty \simeq \mathbb{R}$ and $[[\omega]]$ is a generator. We conclude that $[\omega] \in \mathcal{E}^{m0}_\infty(S_1)$ is a nonzero section. By Lemma 32 we know that $\mathcal{E}^{m0}_\infty$ is of the form $j_!(\mathbb{R}_U)$, so it only can admit a nonzero section if $U = S^1$ and the map $\mathcal{E}^{m0}_\infty \hookrightarrow \mathbb{R}_{S^1}$ is an isomorphism.

□

Theorem 9 (Klingler) *The Euler characteristic of a closed special affine manifold is zero.*

Proof By Lemma 31 and Proposition 4 we know that the map

$$\lambda : \mathcal{B}^{0m}_{c,\infty} \to \mathcal{B}^{0m}_\infty$$

is zero. Proposition 3 implies that in this case the Euler characteristic vanishes. □

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A Some Differential Geometry

A.1 Connections

Given a smooth function $f = (f^1, \ldots, f^m) : M \to \mathbb{R}^m$ and a vector field $X \in \mathfrak{X}(M)$ it makes sense to consider the derivative of f in the direction of X

$$X(f) = (X(f^1), \ldots, X(f^m)) = Df(X).$$

On the other hand, if $\alpha \in \Gamma(E)$ is a section of a vector bundle E, there is no natural way to differentiate α in the direction of a vector field. A connection on a vector bundle E is a rule that prescribes how to differentiate sections of E in the direction of vector fields.

Definition 21 Let $\pi : E \to M$ be a vector bundle. A connection ∇ on E is a linear map

$$\nabla : \mathfrak{X}(M) \otimes \Gamma(E) \to \Gamma(E); \quad (X, \alpha) \mapsto \nabla_X \alpha$$

such that for any smooth function $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\alpha \in \Gamma(E)$ the following two conditions are satisfied:

1.
$$\nabla_{fX} \alpha = f \nabla_X \alpha,$$

2.
$$\nabla_X (f\alpha) = (X(f))\, \alpha + f \nabla_X \alpha.$$

Exercise 8 Show that in the case where $E = M \times \mathbb{R}^m$ is the trivial bundle, the directional derivative described above is a connection on E.

Definition 22 If E is a vector bundle with connection, we will say that a section $\alpha \in \Gamma(E)$ is covariantly constant if $\nabla_X(\alpha) = 0$ for all vector fields $X \in \mathfrak{X}(M)$.

Let us now consider the case $E = TM$ and describe how a connection is expressed in local coordinates $\varphi = (x^1, \ldots, x^m)$. The Christoffel symbols $\Gamma_{ij}^k : M \to \mathbb{R}$ are smooth functions determined by the condition

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k.$$

The connection ∇ is determined by the Christoffel symbols. Given vector fields $X = \sum_i a^i \partial_i$ and $Y = \sum_j b^j \partial_j$ one computes

$$\begin{aligned}\nabla_X Y &= \sum_i a^i \nabla_{\partial_i}\left(\sum_j b^j \partial_j\right) = \sum_{i,j} a^i \nabla_{\partial_i}\left(b^j \partial_j\right)\\ &= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial x^i}\partial_j + b^j \nabla_{\partial_i}\partial_j\right)\\ &= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial x^i}\partial_j + b^j \sum_k \Gamma_{ij}^k \partial_k\right)\\ &= \sum_{i,j} a^i \frac{\partial b^j}{\partial x^i}\partial_j + \sum_{i,j} a^i b^j \sum_k \Gamma_{ij}^k \partial_k\\ &= \sum_k \left(\sum_i a^i \frac{\partial b^k}{\partial x^i}\partial_k + \sum_{i,j} \Gamma_{ij}^k b^j a^i\right)\partial_k.\end{aligned}$$

A Riemannian metric g on a manifold M induces a connection, called the *Levi–Civita Connection*, on the tangent bundle TM.

Definition 23 Let ∇ be a connection on TM. The torsion of ∇ is the function

$$T : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M); \quad (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y].$$

Exercise 9 Show that given vector fields $X, Y \in \mathfrak{X}(M)$, the torsion satisfies

- Linearity with respect to functions:

$$T(fX, Y) = fT(X, Y); \quad T(X, fY) = fT(X, Y).$$

- Skewsymmetry:

$$T(X, Y) + T(Y, X) = 0.$$

The previous exercise implies that one can view the torsion as a tensor

$$T \in \Omega^2(M, TM) = \Gamma(\Lambda^2(T^*M) \otimes TM),$$

defined by

$$T(p)(v, w) = \nabla_X Y(p) - \nabla_Y X(p) - [X, Y](p),$$

for any choice of vector fields X, Y such that $X(p) = v$ and $Y(p) = w$.

Definition 24 A connection on TM is called symmetric if its torsion is zero.

Exercise 10 Show that a connection ∇ is symmetric if and only if for any choice of coordinates, the Christoffel symbols satisfy $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Definition 25 A connection on a riemannian manifold (M, g) is compatible with the metric if

$$X(g(Y, Z)) = g(\nabla_X Y,\ Z) + g(Y,\ \nabla_X Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Theorem 10 (Levi–Civita) *Let (M, g) be a riemannian manifold. There exists a unique symmetric connection ∇ which is compatible with the metric. Moreover, this connection satisfies*

$$\begin{aligned} g(Z, \nabla_Y X) = \frac{1}{2}\,&(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z)). \end{aligned} \tag{1}$$

Proof Any connection compatible with the metric satisfies

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$

Adding the first two equations, subtracting the third and using the symmetry one obtains

$$\begin{aligned} &Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &= g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z) + 2g(Z, \nabla_Y X), \end{aligned}$$

which implies

$$\begin{aligned} g(Z, \nabla_Y X) = \frac{1}{2}\,&(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z)). \end{aligned}$$

Since the metric is nondegenerate, this implies uniqueness. In order to prove existence we define $\nabla_Y X$ to be the unique vector field that satisfies Eq. (1). In order to prove that ∇ defined in this way is a connection, the only nontrivial statement is

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y.$$

For this we compute

$$\begin{aligned} g(Z, \nabla_Y (fX)) = \frac{1}{2}\,&(fXg(Y, Z) + Yg(Z, fX) - Zg(fX, Y) \\ &- g([fX, Z], Y) - g([Y, Z], fX) - g([fX, Y], Z)). \end{aligned}$$

Using the equations

$$Yg\,(Z, fX) = (Yf)\,g\,(Z, X) + fYg\,(Z, X)\,,$$

$$Zg\,(fX, Y) = (Zf)\,g\,(X, Y) + fZg\,(X, Y)\,,$$

$$g\,([fX, Z]\,, Y) = fg\,([X, Z]\,, Y) - (Zf)\,g\,(X, Y)$$

$$g\,([fX, Y]\,, Z) = fg\,([X, Y]\,, Z) - (Yf)\,g\,(X, Z)$$

we obtain

$$\begin{aligned} g\,(Z, \nabla_Y\,(fX)) &= fg\,(Z, \nabla_Y X) + \frac{1}{2}\,(2\,(Yf)\,g\,(Z, X)) \\ &= g\,(Z, f\nabla_Y X + (Yf)\,X)\,. \end{aligned}$$

We leave it as an exercise to the reader to prove that ∇ is symmetric and compatible with the metric. □

The connection described above is called the Levi–Civita connection on (M, g).

A.2 Geodesics and the Exponential Map

Here we will explain the notions of parallel transport and geodesics. It will be convenient to first discuss some natural operations on vector bundles and connections.

Exercise 11 Let ∇ be a connection on $\pi : E \to M$ and $f : N \to M$ a smooth function. Then there exists a unique connection $f^*(\nabla)$ on $f^*(E)$ such that for any $\alpha \in \Gamma(E)$, $X \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ with $Df(p)(X(p)) = Y(f(p))$ the following holds:

$$f^*(\nabla)_X(f^*(\alpha))(p) = \nabla_Y(\alpha)(f(p)). \tag{2}$$

Recall that we say that a section $\alpha \in \Gamma(E)$ of a vector bundle with connection is covariantly constant if $\nabla_X(\alpha) = 0$, for any vector field $X \in \mathfrak{X}(M)$. By imposing this conditions on vector bundles over an interval one obtains the notion of parallel transport along a path.

Proposition 5 *Let ∇ be a connection on a vector bundle $\pi : E \to I$, where $I = [a, b]$ is an interval. Given a vector $v \in E_a$ there exists a unique covariantly constant section $\alpha \in \Gamma(E)$ such that $\alpha(a) = v$. Moreover, the function $P_a^b : E_a \to E_b$ given by $P_a^b(v) = \alpha(b)$ is a linear isomorphism. The function P_a^b is called the parallel transport of the connection ∇.*

Proof Since all vector bundles over an interval are trivializable, we may choose a frame $\{\alpha_1, \ldots, \alpha_k\}$ for E. There exists a one form $\theta \in \Omega^1(I, \mathsf{End}(E))$ such that:

$$\nabla_X(\alpha_i) = \theta(X, \alpha_i).$$

Let us fix $v = \sum_i \lambda_i \alpha_i(a) \in E_a$. A section $\alpha = \sum_i f_i \alpha_i$ is covariantly constant if it satisfies the differential equation

$$\sum_i \nabla_{\partial_t}(f_i \alpha_i) = 0,$$

which is equivalent to

$$\sum_i \frac{\partial f_i}{\partial t}\alpha_i + f_i \theta(\partial_t, \alpha_i) = 0.$$

The Picard–Lindelöf theorem guarantees the existence and uniqueness of a solution of this equation. In order to show that P_a^b is linear it is enough to observe that if α and β are covariantly constant, so is $\alpha + \beta$. It remains to show that P_a^b is an isomorphism. Suppose that $v \in E_a$ is such that $P_a^b(v) = 0$. By symmetry we know that there exists a unique section $\alpha \in \Gamma(E)$ such that $\alpha(b) = 0$. This section is the zero section and we conclude that $v = 0$. □

Definition 26 Let ∇ be a connection on $\pi : E \to M$ and $\gamma : [a, b] \to M$ a smooth curve. The parallel transport along γ with respect to ∇ is the linear isomorphism

$$P_\nabla(\gamma) : E_{\gamma(a)} \to E_{\gamma(b)}; \quad P_\nabla(\gamma)(v) = P_a^b(v),$$

where P_a^b denotes the parallel transport associated with the vector bundle $\gamma^*(E)$ over the interval $I = [a, b]$ with respect to the connection $\gamma^*(\nabla)$.

Lemma 33 *Let $\gamma : [a, c] \to M$ be a curve and $b \in (a, c)$. Set $\mu = \gamma|_{[a,b]}$; $\sigma = \gamma|_{[b,c]}$. Then $P_\nabla(\gamma) = P_\nabla(\sigma) \circ P_\nabla(\mu)$.*

Proof It is enough to observe that if $\alpha \in \Gamma(\gamma^*(E))$ is covariantly constant then $\alpha|_{[a,b]}$ and $\alpha|_{[b,c]}$ are also covariantly constant. □

Exercise 12 Show that parallel transport is parametrization invariant. That is, if ∇ is a connection on $\pi : E \to M$, $\gamma : [a, b] \to M$ is a curve and $\varphi : [c, d] \to [a, b]$ is an orientation preserving diffeomorphism then $P_\nabla(\gamma) = P_\nabla(\gamma \circ \varphi)$.

Definition 27 Let ∇ be a connection on TM. A curve $\gamma : [a, b] \to M$ is called a geodesic if the section $\gamma' \in \Gamma(\gamma^*(TM))$ is covariantly constant with respect to the connection $\gamma^*(\nabla)$.

In local coordinates $\varphi = (x^1, \ldots, x^m)$ where $\gamma = (u_1, \ldots, u_m)$ and ∇ has Christoffel symbols Γ_{ij}^k one has $\gamma'(t) = \sum_i u_i'(t)\partial_i$, and the geodesic equation takes the form

$$\begin{aligned}\gamma^*(\nabla)_{\partial_t}(\gamma'(t)) &= \sum_i \gamma^*(\nabla)_{\partial_t}(u_i'(t)\partial_i)\\ &= \sum_i \left(u_i''(t)\partial_i + u_i'(t)\gamma^*(\nabla)_{\partial_t}\partial_i\right)\\ &= \sum_i \left(u_i''(t)\partial_i + u_i'(t)\sum_j u_j'(t)\nabla_{\partial_j}\partial_i\right)\\ &= \sum_i \left(u_i''(t)\partial_i + u_i'(t)\sum_{j,k} u_j'(t)\Gamma_{ij}^k\partial_k\right).\end{aligned}$$

We conclude that γ is a geodesic precisely when it satisfies the system of differential equations

$$u_i''(t) + \sum_{j,k} u_j'(t)u_k'(t)\Gamma_{kj}^i = 0,\ \forall i. \tag{3}$$

Example 11 On Euclidian space $\mathbb{R}^m$ the Christoffel symbols are $\Gamma_{ij}^k = 0$, and therefore the differential equation for a geodesic is just $u_i''(t) = 0$. We conclude that geodesics in euclidean space are straight lines.

Theorem 12 *Let ∇ be a connection on TM. Given $v \in T_pM$, there exists an interval $(-\epsilon, \epsilon)$ for which there is a unique geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma\,(0) = p$ and $\gamma'(0) = v$.*

Proof Let $\varphi = (x^1, \ldots, x^m)$ be local coordinates such that $\varphi(p) = 0$. We write $\gamma(t) = (u_1(t), \ldots, u_m(t))$ and want to solve the system of equations

$$u_i''(t) + \sum_{j,k} u_j'(t)u_k'(t)\Gamma_{kj}^i = 0.$$

This is a second order ordinary differential equation. The existence and uniqueness of solutions is guaranteed by the Pickard–Lindelöf theorem. □

Definition 28 Let ∇ be a connection on TM and $p \in M$. We define $A_p \subseteq T_pM$ as follows:

$$\begin{aligned}&A_p :=\\ &\{v \in T_pM : \text{there exists a geodesic } \gamma_v : [-1, 1] \to M,\ \text{with } \gamma_v(0) = p \text{ and } \gamma_v'(0) = v\}.\end{aligned}$$

The exponential map is defined by

$$\mathsf{Exp_p} : A_p \to M;\ v \mapsto \gamma_v(1).$$

The proof of the following theorem can be found in any text on riemannian geometry, for example [12].

Theorem 13 *Let ∇ be a connection on TM and $p \in M$. The domain A_p of the exponential map contains an open neighborhood around $0 \in T_pM$. Moreover, the derivative of the exponential map at 0 is the identity and therefore the exponential map is a local diffeomorphism.*

A.3 Curvature and the Chern–Gauss–Bonnet Theorem

The Gauss–Bonnet theorem provides a formula for the Euler characteristic of a closed oriented surface Σ

$$\chi(\Sigma) = \frac{1}{2\pi} \int_\sigma K dA.$$

Here K denotes the Gaussian curvature of Σ associated to a riemannian metric and dA is the volume form determined by the metric and the orientation. This formula is remarkable because while the left hand side evidently depends only on the topology of Σ, the right hand side is a priori a geometric quantity. The extension of this formula to higher dimensions had to wait until the language of differential geometry was developed and Chern [6] proved his generalised version of the Gauss–Bonnet theorem. The first obstruction that needs to be overcome in order to state a correct generelisation is to find a replacement for the gaussian curvature. This is provided by the Riemann curvature tensor.

Definition 29 Let ∇ be a connection on TM. The riemannian curvature of ∇ is the map

$$R : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M),$$

given by

$$R_\nabla(X, Y, Z) := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z.$$

A simple computation shows that this map is $C^\infty(M)$-linear in all the components and skew symmetric in X and Y. Therefore, it defines a tensor

$$R_\nabla \in \Omega^2(M, \mathsf{End}(TM))$$

which is called the Riemann curvature tensor. A connection ∇ is said to be flat if $R_\nabla = 0$.

Exercise 13 Show that if ∇ is the Levi–Civita connection of a riemannian manifold then for any pair of tangent vectors $v, w \in T_pM$, the map

$$R_\nabla(v, w, -) : T_pM \to T_pM$$

is antisymmetric, i.e.

$$\langle R_\nabla(v, w, z), z\rangle = 0.$$

This is one of the Bianchi identities.

Exercise 14 Let $\mathfrak{so}(2n)$ be the Lie algebra of skew symmetric matrices. The Pfaffian polynomial:

$$\mathsf{Pf} : \mathfrak{so}(2n) \to \mathbb{R},$$

is defined by the formula

$$\mathsf{Pf}(A) = \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} sg(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

Show that for $B \in \mathsf{End}(\mathbb{R}^{2n})$ and $A \in \mathfrak{so}(2n)$ the following holds:

(a)

$$\mathsf{Pf}(BAB^t) = \det(B)\mathsf{Pf}(A).$$

(b)

$$\mathsf{Pf}(A)^2 = \det(A).$$

In particular, if $B \in SO(n)$ then

$$\mathsf{Pf}(BAB^{-1}) = \mathsf{Pf}(A).$$

Exercise 15 Let V be a real vector space of dimension $2n$ and $\langle\ ,\ \rangle$ an inner product in V. Let $\mathfrak{so}(V)$ be the space of antisymmetric endomorphisms of V. That is

$$\mathfrak{so}(V) := \{\phi : V \to V : \langle v, \phi(v)\rangle = 0\}.$$

Let $\mathsf{P}(A) = \mathsf{P}(a_{ij}) : \mathfrak{so}(2n) \to \mathbb{R}$ be a polynomial which is invariant under the action of $SO(2n)$ i.e. such that

$$P(BAB^{-1}) = P(A)$$

for all $B \in SO(2n)$. Fix an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ for V. For any $\omega \in \Lambda^2(V^*) \otimes \mathfrak{so}(V)$ define $\omega_{ij} \in \Lambda^2(V^*)$ by the formula

$$\omega_{ij}(v, w) := \langle \omega(v, w)(e_i), e_j\rangle.$$

Show that $\mathsf{P}(\omega) := \mathsf{P}(\omega_{ij}) \in \Lambda(V^*)$ does not depend on the choice of orthonormal basis.

The previous exercise shows that for any riemannian manifold (M, g) of dimension $2n$ there is a well defined form $\mathsf{Pf}(K) \in \Omega^{2n}(M)$ which is defined by

$$\mathsf{Pf}(K)(p) := \mathsf{Pf}(K(p)) \in \Lambda^{2n}(T_p^*M).$$

This differential form is what needs to be integrated over M to obtain the Euler characteristic:

Theorem 14 (Chern–Gauss–Bonnet) *Let (M, g) be a closed oriented riemannian manifold of dimension $d = 2m$ and R the curvature of the Levi–Civita connection. Then*

$$\chi(M) = \left(\frac{1}{2\pi}\right)^n \int_M \mathsf{Pf}(K).$$

Of course, in case the dimension of M is odd, the Euler characteristic vanishes by Poincaré duality.

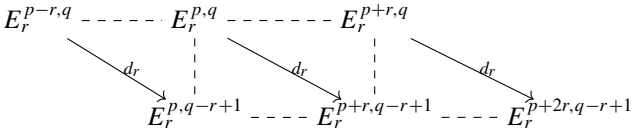
B Spectral Sequences

This section contains a brief introduction to spectral sequences.

Definition 30 A cohomologically graded spectral sequence of real vector spaces $E = \{E_r\}$ is a sequence of bigraded vector spaces $E_r = \oplus E_r^{p,q}$ together with differentials of degree $(r, -r + 1)$

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $d_r^2 = 0$ and $E_{r+1} = H^*(E_r)$.



(4)

A morphism between spectral sequences $f : E \to F$ is a sequence of linear maps $f_r : E_r \to F_r$ compatible with the differentials such that f_{r+1} is the morphism induced by f_r, i.e.

$$f_{r+1} = H^*(f_r) : E_{r+1} = H^*(E_r) \to F_{r+1} = H^*(F_r).$$

Spectral sequences form a category.

We usually think of the terms in a spectral sequence as pages in a book where the index r denotes the page and (p, q) are coordinates in that page.

Let $Z_1 = \ker(d_1)$ and $B_1 = \mathrm{Im}(d_1)$. Since $E_2 = H^*(E_1) = Z_1/B_1$, we may regard the kernel and image of d_2 as quotients $\ker(d_2) = Z_2/B_1$ and $\mathrm{Im}(d_2) = B_2/B_1$, whence $B_1 \subset B_2 \subset Z_2 \subset Z_1$. By induction we have $E_{r+1} = Z_r/B_r$, $\ker(d_{r+1}) = Z_{r+1}/B_r$, $\mathrm{Im}(d_{r+1}) = B_{r+1}/B_r$ and $B_r \subset B_{r+1} \subset Z_{r+1} \subset Z_r$. This chain of inclusions allows us to define

$$B_\infty = \cup B_r, \qquad Z_\infty = \cap Z_r.$$

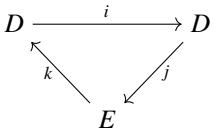
Clearly $B_\infty \subset Z_\infty$.

Definition 31 The term at infinity of a spectral sequence is $E_\infty = B_\infty/Z_\infty$.

A simple example of a spectral sequence is a cochain complex $d : C^\bullet \to C^{\bullet+1}$. Taking $E_1 = C$ with the differential of the complex and $E_2 = H^*(C)$ with zero differential. Notice that $E_r = E_2$ for $r \geq 2$, therefore the sequence stabilizes at the second page making $E_\infty = E_2$.

Another source of examples of spectral sequences are exact couples.

Definition 32 An exact couple is a couple of bigraded vector spaces D, E with linear maps i, j, k such that the following diagram is exact:



Lemma 34 *Given an exact couple $\{D, E, i, j, k\}$ we can produce another exact couple as follows. Let $d = jk$, then $d^2 = jkjk = 0$ which makes d a differential on E, we will denote classes in homology by overlines. Let $E' = H^*(E)$ and $D' = i(D)$, i' the restriction of i to D', $j'(i(x)) = \overline{j(x)}$ and k' the map induced by k. Then $\{D', E', i', j', k'\}$ is an exact couple.*

Proof The proof is elementary. First we check that j' and k' are well defined:

- For j' suppose that $i(x) = i(y)$, which means that $x - y \in \ker(i) = \mathrm{Im}(k)$, so $x - y = k(e)$ for some $e \in E$. Applying j we get $j(x - y) = jk(e) = d(e)$, therefore $\overline{j(x)} = \overline{j(y)}$.
- For k' we want to check that $k'(\bar{e}) = k(e) \in D' = \ker(j)$. Since e defines a homology class we have $d(e) = jk(e) = 0$, so $k(e) \in \ker(j)$.

Now we go for exactness:

- At $\cdots \xrightarrow{k'} D' \xrightarrow{i'} \cdots$ Clearly $i'(k'(\bar{e})) = ik(e) = 0$. Now take $i(x) \in D'$ such that $i'(i(x)) = 0$, by exactness in D there is $e \in E$ such that $k(e) = i(x)$, so $d(e) = jk(e) = ji(x) = 0$, whence e defines a class in homology and $k'(\bar{e}) = k(e) = i(x)$.

- At $\cdots \xrightarrow{i'} D' \xrightarrow{j'} \cdots$ For $i(x) \in D'$ we have $j'(i'(i(x))) = \overline{j(i(x))} = 0$. Now suppose $j'(i(x)) = \overline{j(x)} = 0$, then $j(x) = d(e) = jk(e)$ for some $e \in E$, meaning that $x - k(e) \in \ker(j) = \mathrm{Im}(i)$. Let $y \in D$ such that $i(y) = x - k(e)$, then $i(x) = i(x) - ik(e) = ii(y) \in \mathrm{Im}(i')$.
- At $\cdots \xrightarrow{j'} E' \xrightarrow{k'} \cdots$ Take $i(x) \in D'$, then $k'(j'(i(x))) = k'(\overline{j(x)}) = kj(x) = 0$. Now if $k'(\bar{e}) = k(e) = 0$, then there is $x \in D$ with $j(x) = e$, then clearly $j'(i(x)) = \overline{j(x)} = \bar{e}$.

□

Consider now an initial exact couple $\{D_0, E_0, i_0, j_0, k_0\}$ such that i_0, j_0, k_0 have degrees $(-1, 1)$, $(0, 0)$, $(0, 1)$ respectively. Applying the previous construction consecutively we get a sequence of exact couples $\{D_r, E_r, i_r, j_r, k_r\}$ with degrees $(-1, 1)$, $(r, -r)$, $(0, 1)$. Notice that $d_r = j_r k_r$ is a differential on E_r of degree $(r, -r + 1)$, and by construction $E_r = H^*(E_{r-1})$, making $\{E_r\}$ a spectral sequence.

Given a filtered cochain complex we can produce an exact couple which in turn gives rise to a spectral sequence. Suppose $\{C^\bullet, d\}$ is a cochain complex with a decreasing filtration $\cdots, F^pC \supset F^{p+1}C \supset \cdots$ of subcomplexes. The graded complex associated to this filtration is actually bigraded, let E_0 denote the graded complex, then $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$. The differential on E_0 is $\bar{d} : E_0^{p,q} \to E_0^{p,q+1}$ which is induced by d. Notice that $\bar{d}$ has degree $(0, 1)$. Now consider the short exact sequence of cochain complexes

$$0 \to F^{p+1}C^\bullet \xrightarrow{i} F^pC^\bullet \xrightarrow{j} E_0^{p,\bullet} \to 0$$

By the snake lemma we get a long exact sequence

$$\cdots \to H^n(F^{p+1}C^\bullet) \xrightarrow{i_*} H^n(F^pC^\bullet) \xrightarrow{j_*} H^n(E_0^{p,\bullet}) \xrightarrow{k} H^{n+1}(F^{p+1}C^\bullet) \to \cdots$$

Finally let us define $D_1^{p,q} = H^{p+q}(F^{p+1}C^\bullet)$ and $E_1^{p,q} = H^{p+q}(E_0^{p,\bullet})$. Notice that with this convention the maps i_*, j_*, k have degrees $(-1, 1)$, $(1, -1)$ and $(0, 1)$. Clearly $\{D_1, E_1, i_*, j_*, k\}$ is an exact couple, thus we have a spectral sequence associated to the filtration.

We will give an explicit formula for the terms of $\{E_r\}$. We start by defining the cycles up to filtration, this is, for $r \geq 0$

$$A_r^{p,q} = \{x \in F^pC^{p+q} \mid d(x) \in F^{p+r}C^{p+q+1}\} \tag{5}$$

Lemma 35 *The terms E_r of the spectral sequence are*

$$E_r^{p,q}C = \frac{\left(A_r^{p,q} + F^{p+1}C^{p+q}\right)}{\left(d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}\right)} \tag{6}$$

Proof First notice that Eq. (6) for $r = 0$ yields $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$. We will prove that projection $j : F^pC \to F^pC/F^{p+1}C$ induces a surjective map $\bar{j} : A_r^{p,q} + F^{p+1}C^{p+q} \to Z_{r-1}^{p,q}$. Furthermore, we will see that $\bar{j}(d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}) = B_{r-1}^{p,q}$ and the inverse image of $B_{r-1}^{p,q}$ is precisely $d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}$. Thus $\bar{j}$ induces an isomorphism

$$\frac{A_r^{p,q} + F^{p+1}C^{p+q}}{d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}} \cong \frac{Z_{r-1}^{p,q}}{B_{r-1}^{p,q}} = E_r^{p,q}.$$

For this purpose we prove the following equations:

$$Z_r = k^{-1}(\mathrm{Im}(i_r)), \qquad B_r = j(\ker(i_r)). \tag{7}$$

For the first equation take $z \in Z_r$. $d_r(\bar{z}) = j_r k_r(\bar{z}) = 0$ is equivalent to $k_r(\bar{z}) \in \ker(j_r) = \mathrm{Im}(i_r)$. Since $k_r(z)$ is induced by the map k, we have $k_r(\bar{z}) = k(z)$, which means $k_r(\bar{z}) \in \mathrm{Im}(i_r)$ if and only if $z \in k^{-1}(\mathrm{Im}(i_r))$.

Now take an element $j_r k_r(z) \in B_r$, then $k_r(z) \in \mathrm{Im}(k_r) = \ker(i_r)$. j_r is just a restriction of the map j, hence $B_r = j(\ker(i_r))$.

Let us define the map $\bar{j}$. Take $x \in A_r^{p,q} + F^{p+1}C^{p+q}$ and consider its image $j(x) \in E_0^{p,q}$. Since $d(x) \in F^{p+r}C$ we have $d(j(x)) = 0$, therefore $j(x)$ defines an element in $H^*(E_0^{p,q})$ which we denote $\bar{x}$. Notice that $k(\bar{x}) = \overline{d(x)} \in H^*(F^{p+1}C)$, it is not always zero since $d(x)$ is not necessarily a boundary in the complex $F^{p+1}C$, furthermore, $k(\bar{x}) \in \mathrm{Im}(i_{r-1})$, whence $\bar{x} \in k^{-1}(\mathrm{Im}(i_{r-1})) = Z_{r-1}$.

$\bar{j}$ is surjective: take an element $w \in Z_{r-1}^{p,q}$, hence $k(w) = i_{r-1}(y)$ for some $y \in H^{p+q+1}(F^{p+r}C)$, we will also denote a representative of y by y. Next suppose that $j(x_1)$ is a representative of w, this means that $d(x_1) = k(w)$, therefore $d(x_1)$ and y represent the same class in $H^{p+q+1}(F^pC)$, hence there is an $x_2 \in F^pC^{p+q}$ such that $d(x_1) - y = d(x_2)$. Taking $x = x_1 + x_2 \in A_r^{p,q} + F^{p+1}C^{p+q}$ we get that $j(x)$ represents w, and $d(x) = y$, making $\bar{j}(x) = w$.

Let us check that $\bar{j}(d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}) \subset B_{r-1}^{p,q}$. Notice that the components in $F^{p+1}C^{p+q}$ maps to zero, therefore it is enough to check that $\bar{j}d(A_{r-1}^{p-r+1,q+r-2}) \subset B_{r-1}^{p+q}$. Let $d(w) \in d(A_{r-1}^{p-r+1,q+r-2})$, which means that $w \in F^{p-r+1}C^{p+q-1}$. We have that $d(w)$ is not necessarily a boundary in F^pC^{p+q}, but becomes a boundary after applying the inclusion $F^pC^{p+q} \subset F^{p-r+1}C^{p+q}$, which means that its homology class goes to zero in $H^{p+q}(F^{p-r+1}C)$. We have proved that $\bar{j}(d(w)) \in j(\ker(i_{r-1})) = B_{r-1}$.

Finally let us check that $\bar{j}^{-1}(B_{r-1}) = d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}$. Suppose $\bar{j}(x) \in B_{r-1}$, this is, $\bar{j}(x) = j(u)$ for $u \in \ker(i_{r-1})$. Then u must be a boundary in $F^{p-r+1}C$, this is, $u = d(z)$. Next notice that $j(x - d(z)) = 0$, so $x - d(z) \in F^{p+1}C$ and $x = d(z) + x - d(z) \in d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1}C^{p+q}$. □

Corollary 4 *The term at infinity of the spectral sequence is*

$$E_{\infty}^{p,q}C = \frac{\left(A_{\infty}^{p,q} + F^{p+1}C^{p+q}\right)}{\left(d(A_{\infty})^{p,q} + F^{p+1}C^{p+q}\right)}, \tag{8}$$

where

$$A_{\infty}^{p,q} = \cap_{r\geq 1} A_r^{p,q} \quad \text{and} \quad \left(d(A_{\infty})^{p,q}\right) = \cup_{r\geq 1} d(A_{r-1}^{p-r+1,q+r-2}). \tag{9}$$

Let C be a cochain complex with a decreasing filtration of subcomplexes $\cdots \supset F^pC \supset F^{p+1}C \supset \cdots$. We denote by E_rC the spectral sequence associated to the filtered complex. The cohomology of the complex may also be filtered by taking $F^pH^*(C) = \mathrm{Im}(H^*(F^pC) \to H^*(C))$, whence we have a second spectral sequence $E_rH^*(C)$. The following theorem establishes a relation between both sequences.

Theorem 15 *If C is a filtered complex such that $C = \cup_p F^pC$ and for every n there exists an m such that $F^mC^n = 0$, then $E_{\infty}^{p,q}C = E_0^{p,q}H^*(C)$.*

Proof We start by considering the induced filtrations on Z and B, the cycles and boundaries of C:

$$F^pZ^s = F^pC \cap \ker\{d : C^s \to C^{s+1}\}, \qquad F^pB^s = F^pC \cap \mathrm{Im}(d : C^{s-1} \to C^s),$$

then $H(F^pC) = F^pZ/F^pB$.

We will compute $E_0^{p,q}H^*(C)$. Recall that $E_0^{p,q}H^*(C) = F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C)$ where $F^pH(C) = \mathrm{Im}(H(F^pC) \to H(C))$. Notice that $F^pH^{p+q}(C) = (F^pZ + B)/B$, therefore we have

$$E_0^{p,q}H^*(C) = \frac{F^pZ + B}{F^{p+1}Z + B} = \frac{F^pZ}{F^pZ \cap (F^{p+1}Z + B)}.$$

Furthermore, we have the equality $F^pZ \cap (F^{p+1}Z + B) = F^pZ \cap (F^{p+1}C + F^pB)$. Indeed, let $x \in F^pZ \cap (F^{p+1}Z + B)$, then $x = a + d(b)$ with $a \in F^{p+1}Z$ and $d(b) \in B$. Clearly $a \in F^{p+1}C$ and $d(b) = x - a \in F^pC$, so $x = a + d(b) \in F^{p+1}C + F^pB$. Similarly, for $x \in F^pZ \cap (F^{p+1}C + F^pB)$ we have $x = a + d(b)$ with $a \in F^{p+1}C$ and $d(b) \in F^pB \subset B$. We have that $0 = d(x) = d(a) + d^2(b) = d(a)$, so $a \in F^{p+1}Z$. We have proven that

$$E_0^{p,q}H^*(C) = \frac{F^pZ}{F^pZ \cap (F^{p+1}C + F^pB)} = \frac{F^pZ + F^{p+1}C}{F^pB + F^{p+1}C}$$

Recall the cycles up to filtration defined in (5), notice that under our hypothesis, for r large enough we have $F^pZ^* = A_r^{p,*}$, whence

$$F^pZ^* + F^{p+1}C^* = A_r^{p,*} + F^{p+1}C^* = A_{\infty}^{p,*} + F^{p+1}C^*.$$

On the other hand, for any $d(b) \in F^p B$ we must have some s such that $b \in F^s C$. So $d(b) \in d(A_{p-s}^{s,p+q-s-1}) \subset d(A_\infty)^{p,q}$, therefore

$$F^p B^* + F^{p+1} C^* = d(A_\infty)^{p,*} + F^{p+1} C^*.$$

Combining our previous findings we get

$$E_0^{p,q} H^*(C) = \frac{\left(A_\infty^{p,q} + F^{p+1} C^{p+q}\right)}{\left(d(A_\infty)^{p,q} + F^{p+1} C^{p+q}\right)} = E_\infty^{p,q} C.$$

□

The following lemma provides an alternate formula for the terms of the spectral sequence that is more convenient for computational purposes.

Lemma 36 *The terms of the spectral sequence associated to a filtered complex $F^\bullet C$ are:*

$$E_r^{p,q} C = \frac{A_r^{p,q}}{d(A_{r-1}^{p-r+1,q+r-2}) + A_{r-1}^{p+1,q-1}}, \tag{10}$$

where $A_r^{p,q}$ is as in (5).

The terms at infinity are

$$E_\infty^{p,q} C = \frac{A_\infty^{p,q}}{d(A_\infty)^{p,q} + A_\infty^{p+1,q-1}}, \tag{11}$$

where $A_\infty^{p,q}$ and $d(A_\infty)^{p,q}$ are as in (9).

Proof We have the equation

$$A_r^{p,q} \cap F^{p+1} C^{p+q} = \{x \in F^{p+1} C^{p+q} \mid d(x) \in F^{p+r} C^{p+q+1}\} = A_{r-1}^{p+1,q-1}.$$

Since $d(A_{r-1}^{p-r+1,q+r-2}) \subset A_r^{p,q}$, the isomorphism follows

$$\frac{A_r^{p,q} + F^{p+1} C^{p+q}}{d(A_{r-1}^{p-r+1,q+r-2}) + F^{p+1} C^{p+q}} = \frac{A_r^{p,q}}{d(A_{r-1}^{p-r+1,q+r-2}) + A_r^{p,q} \cap F^{p+1} C^{p+q}}.$$

Equation (11) is proved in a similar fashion. □

Example 16 Given a complex C, a simple filtration is the so called bête filtration:

$$(F^p C)^n = \begin{cases} 0 & \text{if } n < p \\ C^n & \text{if } n \geq p \end{cases}$$

In this case the spectral sequence converges to the cohomology of the complex.

Example 17 As a more interesting example we compute the first terms of the spectral sequence associated to the vertical filtration of a double complex. Suppose Ω is a first quadrant double complex:

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \\ \Omega^{0,1} & \xrightarrow{d_H^{0,1}} & \Omega^{1,1} & \longrightarrow & \cdots \\ {\scriptstyle d_V^{0,0}}\uparrow & & {\scriptstyle d_V^{1,0}}\uparrow & & \\ \Omega^{0,0} & \xrightarrow{d_H^{0,0}} & \Omega^{1,0} & \longrightarrow & \cdots \end{array}$$

The chain complex is $C^n = \oplus_{i+j=n}\Omega^{i,j}$ with differential $d^{i,j} = d_H^{i,j} + d_V^{i,j}$. The filtration is then

$$F^r C^{p+q} = \bigoplus_{\substack{i+j=p+q \\ j\geq r}} \Omega^{i,j}.$$

The zeroth term of the sequence is

$$E_0^{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{\Omega^{q,p} \oplus \Omega^{q-1,p+1} \oplus \cdots \oplus \Omega^{p+q,0}}{\Omega^{q-1,p+1} \oplus \cdots \oplus \Omega^{p+q,0}} = \Omega^{q,p}$$

Using (10) we compute the first term of the sequence:

$$A_1^{p,q} = \bigoplus_{\substack{i+j=p+q \\ j\geq p+1}} \Omega^{i,j} \oplus \ker(d_H^{q,p})$$

$$A_0^{p+1,q-1} = \bigoplus_{\substack{i+j=p+q \\ j\geq p+1}} \Omega^{i,j}$$

$$A_0^{p,q-1} = \bigoplus_{\substack{i+j=p+q-1 \\ j\geq p}} \Omega^{i,j}$$

Notice that

$$d(A_0^{p,q-1}) + A_0^{p+1,q-1} = \bigoplus_{\substack{i+j=p+q \\ j\geq p+1}} \Omega^{i,j} \oplus \mathrm{Im}(d_H^{q-1,p})$$

therefore

$$E_1^{p,q} = \frac{\left(\bigoplus_{\substack{i+j=p+q \\ j\geq p+1}} \Omega^{i,j}\right) \oplus \ker(d_H^{q,p})}{\left(\bigoplus_{\substack{i+j=p+q \\ j\geq p+1}} \Omega^{i,j}\right) \oplus \mathrm{Im}(d_H^{q-1,p})} = \frac{\ker(d_H^{q,p})}{\mathrm{Im}(d_H^{q-1,p})}.$$

So the page E_1 of the spectral sequence is just the horizontal cohomology of the double complex which we denote $E_1^{p,q} = H_H^q(\Omega^{\bullet,p})$. Recall that the differential on E_1 is such that

$$d_1 : E_1^{p,q} = H_H^q(\Omega^{\bullet,p}) \to E_1^{p+1,q} = H_H^q(\Omega^{\bullet,p+1}),$$

therefore the second term of the spectral sequence is

$$E_2^{p,q} = H_V^p H_H^q(\Omega).$$

If we use the horizontal filtration instead, the computation follows with minor changes. The filtration is

$$F^r C^{p+q} = \bigoplus_{\substack{i+j=p+q \\ i \geq r}} \Omega^{i,j}.$$

The zeroth term is $E_0^{p,q} = \Omega^{p,q}$, the first term is the vertical cohomology $E_1^{p,q} = H_V^q(\Omega^{p,\bullet})$ and the second term is $E_2^{p,q} = H_H^p H_V^q(\Omega)$.

C Sheaves

C.1 Basic Definitions and Results

All spaces considered in this section are locally compact.

Definition 33 Let X be a topological space. A presheaf of vector spaces F over X consists of the following data:

- For every open set $U \subset X$ there is a vector space $F(U)$.
- For every inclusion $U \subset V$ there is a linear map $F(V) \to F(U)$ called a restriction map.

The elements of $F(U)$ are called sections of F over U. If $U \subset V$ and $s \in F(V)$, we often denote the image of the restriction of s to U by $s|_U$. These restrictions must satisfy the following conditions:

- The restriction $F(U) \to F(U)$ must be the identity.
- If $U \subset V \subset W$ and $s \in F(W)$, then $(s|_V)|_U = s|_U$.

In the previous definition we may change the terms "vector spaces" and "linear maps" for objects and morphisms in any category. In fact, another way to define a presheaf is as follows: given a topological space X, we have the category $O(X)$ whose objects are open subsets of X and whose morphisms are inclusions. If C is any other category, a presheaf over X with values in C is a contravariant functor

from $O(X)$ to C. We are mostly interested in the case where $C = \text{Vect}$, the category of real vector spaces.

Definition 34 Let F be a presheaf over X and $x \in X$. The stalk of F at x, denoted F_x, is the direct limit

$$F_x = \varinjlim_{U} F(U),$$

where the limit is taken over all open sets containing x. If $s \in F(U)$, the image image of s under the projection $F(U) \to F_x$ is denoted s_x and is called the germ of s at x.

Definition 35 A sheaf over X is a presheaf F that satisfies the following conditions:

- If $\{U_\alpha\}_\alpha$ is an open covering of an open set U and s, t are sections over U such that $s|_{U_\alpha} = t|_{U_\alpha}$ for every α, then $s = t$.
- If $\{U_\alpha\}_\alpha$ is an open covering of an open set U and there are sections $s_\alpha \in F(U_\alpha)$ for every α such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for each pair α, β, then there is a section $s \in F(U)$ such that $s|_{U_\alpha} = s_\alpha$.

If F, G are presheaves over X with values in the same category, a morphism $\varphi : F \to G$ is a family of morphisms $\varphi_U : F(U) \to G(U)$ for every open set $U \subset X$. These morphisms must be compatible with the restriction morphisms, this is, if $U \subset V$, the following diagram must be commutative:

$$\begin{array}{ccc} F(V) & \xrightarrow{\varphi_V} & G(V) \\ \downarrow & & \downarrow \\ F(U) & \xrightarrow{\varphi_U} & G(U) \end{array}$$

Categorically, a morphism of presheaves is a natural transformation $F \to G$. A morphism of sheaves is just a morphism as presheaves.

We will denote the category of presheaves over X by $PSh(X)$ and the category of sheaves by $Sh(X)$. Notice that $Sh(X)$ is a full subcategory of $PSh(X)$. Given $x \in X$, taking the stalk of a sheaf at x is a functor

$$(-)_x : PSh(X) \to \text{Vect}.$$

If $U \subset X$ is an open set, we define the functor

$$\Gamma(U; -) : PSh(X) \to \text{Vect}$$

by $\Gamma(U; F) = F(U)$. This functor is called taking sections in U. In particular we call $\Gamma(X; -)$ the global sections functor.

Proposition 6 *If F is a presheaf over X, then there is a sheaf F^+ over X and a morphism of presheaves $\theta : F \to F^+$ unique up to isomorphism with the property*

that, for any other sheaf G and morphism $\lambda : F \to G$, there is a unique morphism $\widehat{\lambda} : F^+ \to G$ such that $\lambda = \widehat{\lambda} \circ \theta$.

In other words, the map

$$Hom_{Sh(X)}(F^+, G) \to Hom_{PSh(X)}(F, G)$$

induced by θ is an isomorphism. This means that the functor $PSh(X) \to Sh(X)$ given by $F \mapsto F^+$ is a left adjoint of the forgetful functor.

The sheaf F^+ is called the sheaf associated to the presheaf F.

Proof We define $F^+(U)$ as the set of functions $\sigma : U \to \bigsqcup_{x \in U} F_x$ such that:

1. $\sigma(x) \in F_x$.
2. For every $x \in U$ there is a neighbourhood $V \subset U$ of x and a section $s \in F(V)$ such that for every $y \in V$, we have $\sigma(y) = s_y$.

The first condition states that σ is a section with respect to the natural projection $\bigsqcup_{x \in U} F_x \to U$. The morphism $\theta : F \to F^+$ is such that for $s \in F(U)$ and $x \in U$, $\theta(s)(x) = s_x$. If we endow $\bigsqcup_{x \in U} F_x$ with the finest topology such that all the sections of the form $\theta(s)$ are continuous, then the second condition on σ is just a requirement of continuity. The properties are readily checked. □

Given a sheaf F over X and a continuous function $h : X \to Y$, we are interested in the following two ways of transferring the data of F to Y via h:

Definition 36 The direct image of F is a sheaf on Y defined as follows: if $W \subset Y$ is an open set, we set $h_* F(W) := F(h^{-1}(W))$.

Definition 37 Given a section $s \in F(U)$, the support of s, denoted $supp(s)$, is the closure $\overline{U - Z}$ where Z is the union of all open sets $V \subset U$ such that $s|_V = 0$. The direct image with compact support is a sheaf $h_!(F)$ on Y defined as follows: for an open set $W \subset Y$, we set:

$$h_! F(W) = \{s \in F(h^{-1}(W)) : h|_{supp(s)} : supp(s) \to W \text{ is proper}\}.$$

Given a sheaf G over Y, we can also transfer the data of G to X via h.

Definition 38 The sheaf $h^{-1}G$ over X called the inverse image of G is defined as follows: if U is open in X, then

$$U \mapsto \varinjlim_{V} G(V),$$

where V ranges over the open neighbourhoods of $h(U)$. This procedure produces a presheaf, taking its associated sheaf we get the inverse image $h^{-1}G$. In the case where $i : Z \hookrightarrow X$ is a subspace and F is a sheaf on X, we will denote the inverse image $i^{-1}F = F|_Z$, the restriction of F to Z. Also, when taking sections of F over

Z we write $\Gamma(Z; F) = \Gamma(Z; i^{-1}F)$. The sections of F over Z can also be obtained as a direct limit

$$\Gamma(Z; F) = \varinjlim_{U} \Gamma(U; F), \tag{12}$$

where U ranges over all open sets containing Z.

When taking sections with compact support of a sheaf over a space, we will write Γ_c, this is,

$$\Gamma_c(X, F) = \{s \in \Gamma(X, F) \quad \big| \quad \text{supp}(s) \text{ is compact.}\}.$$

Proposition 7 *Let $h : Y \to X$ be a continuous function and G a sheaf over Y. Then for every $x \in X$, the are canonical morphisms*

$$\alpha : (h_*G)_x \to \Gamma(h^{-1}(x); G|_{h^{-1}(x)}) \quad \textit{and} \quad \beta : (h_!G)_x \to \Gamma_c(h^{-1}(x); G|_{h^{-1}(x)}) \tag{13}$$

The morphism β is actually an isomorphism.

Proof β is injective: Take $s_x \in (h_!G)_x$ such that $\beta(s_x) = 0$. A representative of s_x is a section $s \in \Gamma(h^{-1}(U); G)$ such that $x \in U$ and $h|_{supp(s)} : \text{supp}(s) \to U$ is proper. $\beta(s_x) = 0$ implies that $supp(s)$ and $h^{-1}(x)$ are disjoint, therefore $x \notin h(\text{supp}(s))$. Since $h(\text{supp}(s))$ is closed, there is a neighbourhood of x where the section vanishes, which means that $s_x = 0$.

β is surjective: Take $s \in \Gamma_c(h^{-1}(x); G|_{h^{-1}(x)})$ and name $K = \text{supp(s)}$. By (12), there is a neighbourhood U of K and a section t over U such that $t|_K = s|_K$. Restricting the open if necessary, we may assume that $t|_{U\cap h^{-1}(x)} = s|_{U\cap h^{-1}(x)}$. Next take a neighbourhood V of K such that $K \subset V \subset \bar{V} \subset U$. We have that $x \notin h(\bar{V} \cap \text{supp}(t) \setminus V)$, there is an open neighbourhood W of x such that $h^{-1}(W) \cap \bar{V} \cap \text{supp}(t) \subset V$. Let us define $\widetilde{s} \in \Gamma(h^{-1}(W); G)$ as

$$\widetilde{s}|_{h^{-1}(W)\setminus(\text{supp}(t)\cap\bar{V})} = 0, \qquad \widetilde{s}|_{h^{-1}(W)\cap V} = t|_{h^{-1}(W)\cap V}.$$

The section $\widetilde{s}$ maps to s via β. □

C.2 Cohomology of Sheaves

All sheaves considered in this section are over a space X.

Definition 39 Let O_X be a sheaf of rings with unit over X, we call (X, O_X) a ringed space. A sheaf A is a sheaf of modules over O_X if for every open $U \subset X$, $\Gamma(U; A)$ is a module over $\Gamma(U; O_X)$ and for every inclusion of open sets $V \subset U$, the restriction map is a morphism of $\Gamma(U; O_X)$-modules.

Consider a short exact sequence of sheaves of O_X-modules $0 \to A \to B \to C \to 0$. The functor $\Gamma(X; -)$ is left exact, i.e. the sequence $0 \to \Gamma(X; A) \to \Gamma(X; B) \to \Gamma(X; C)$ is exact. The map $\Gamma(X; B) \to \Gamma(X; C)$ is not surjective in general, however, there are functors $\{H^k(X; -)\}_{k\geq 0}$ that complete the sequence, this means that there is a long exact sequence

$$0 \to H^0(X; A) \to H^0(X; B) \to H^0(X; C) \to \\ \to H^1(X; A) \to H^1(X; B) \to H^1(X; C) \to \cdots \quad (14)$$

where $H^0(X; -) = \Gamma(X; -)$. The groups $H^k(X; A)$ are called the cohomology groups of X with coefficients in A.

In order to construct the cohomology groups, we will use injective resolutions of sheaves. A sheaf I is said to be injective if for every injective morphism $i : A \hookrightarrow B$ and every morphism $\phi : A \to I$ there is an extension of ϕ to B. That is, a morphism $\widetilde{\phi} : B \to I$ such that $\widetilde{\phi} \circ i = \phi$.

A resolution of A is a sequence of sheaves $\{I^k\}_{k\geq 0}$ and morphisms $\{d^k\}_{k\geq 0}$ such that

$$0 \to A \longrightarrow I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots$$

is exact. An injective resolution is one where every sheaf I^k is injective. We will denote resolutions by $0 \to A \xrightarrow{d^\bullet} I^\bullet$.

Proposition 8 *Every sheaf of O_X-modules has an injective resolution. This is often stated as the existence of enough injective objects in the category of sheaves of O_X-modules over X.*

Now let us take an arbitrary sheaf A and an injective resolution $0 \to A \xrightarrow{d^\bullet} I^\bullet$. Consider the sequence obtained when we remove A

$$0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

Applying the functor $\Gamma(X; -)$ we get a complex

$$0 \to \Gamma(X; I^0) \xrightarrow{\Gamma(X;d^0)} \Gamma(X; I^1) \xrightarrow{\Gamma(X;d^1)} \Gamma(X; I^2) \longrightarrow \cdots$$

The homology groups of this complex are the cohomology groups of X with values in A:

$$H^k(X; A) = \frac{\ker(\Gamma(X; d^k))}{\mathrm{Im}(\Gamma(X; d^{k-1}))}$$

In particular we have $H^0(X; A) = \ker(\Gamma(X; d^0)) = \Gamma(X; A)$ since $\Gamma(X; -)$ is left exact.

The following proposition states that the cohomology groups depend only on A and not on the choice of injective resolution.

Proposition 9 *If* $0 \to A \xrightarrow{d^\bullet} I^\bullet$ *and* $0 \to A \xrightarrow{d'^\bullet} I'^\bullet$ *are injective resolutions of A, then there is a morphism of resolutions* $\varphi^\bullet : I^\bullet \to I'^\bullet$ *lifting the identity*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \xrightarrow{d^0} & I^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle Id_A} & & \downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \varphi^1} & & \\ 0 & \longrightarrow & A & \longrightarrow & I'^0 & \xrightarrow[d'^0]{} & I'^1 & \longrightarrow & \cdots \end{array}$$

This morphism induces an isomorphism at the level of homology.

Proposition 10 *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of sheaves. If* $0 \to A \xrightarrow{d_A^\bullet} I_A^\bullet$ *and* $0 \to C \xrightarrow{d_C^\bullet} I_C^\bullet$ *are injective resolutions for A and C respectively, then* $0 \to B \longrightarrow I_A^\bullet \oplus I_C^\bullet$ *is an injective resolution for B. Furthermore,* $0 \to I_A^\bullet \to I_A^\bullet \oplus I_C^\bullet \to I_C^\bullet \to 0$ *is a short exact sequence of resolutions.*

Applying the snake lemma to the sequence above one obtains (14).

C.3 The Functors Rh_*, $Rh_!$ and the Induced Spectral Sequence

The procedure applied to construct sheaf cohomology may be generalized to arbitrary left exact functors. Let A be a sheaf over X and $0 \to A \xrightarrow{d^\bullet} I^\bullet$ an injective resolution of A. Removing A we obtain a (non-exact) sequence

$$0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

Now let $\mathcal{F} : Sh(X) \to C$ be a left exact functor into an abelian category C. Applying $\mathcal{F}$ to the previous sequence we obtain a complex

$$0 \to \mathcal{F}(I^0) \xrightarrow{\mathcal{F}(d^0)} \mathcal{F}(I^1) \xrightarrow{\mathcal{F}(d^1))} \mathcal{F}(I^2) \longrightarrow \cdots \tag{15}$$

The nth derived functor of $\mathcal{F}$, denoted $R^n\mathcal{F}$, is the nth cohomology of the previous complex. Left exactness of $\mathcal{F}$ implies that $R^0\mathcal{F} = \mathcal{F}$.

If $h : X \to Y$ is a continuous function, then h_* and $h_!$ are left-exact functors $Sh(X) \to Sh(Y)$. In particular we have the derived functors Rh_* and $Rh_!$.

Remark 2 The derived functors Rh_* and $Rh_!$ may be computed using the spectral sequence associated to the bête filtration explained in Example 16 for the complex (15).

Proposition 11 *The morphisms (13) from Proposition 7 may be extended to morphisms*

$$\alpha^p : (R^p h_*(G))_x \to H^p(h^{-1}(x); G_{h^{-1}(x)})$$

and

$$\beta^p : (R^p h_!(G))_x \to H_c^p(h^{-1}(x); G_{h^{-1}(x)}).$$

As before, β^p is an isomorphism.

This result is known as the proper base change theorem and may be consulted in [10].

Remark 3 The morphism α (β) induces a morphism between the spectral sequence computing the derived functors of h_* (respectively $h_!$) over a point x and the spectral sequence computing the sheaf cohomology $H^p(h^{-1}(x); G_{h^{-1}(x)})$ (respectively $H_c^p(h^{-1}(x); G_{h^{-1}(x)})$)

C.4 Acyclic, Flabby, Soft, Fine and Constructible Sheaves

Injective sheaves are inadequate for computational purposes of sheaf cohomology. In this section we introduce sheaves that are better suited for the task. Once again, all sheaves considered are over X. A sheaf L is called acyclic if $H^k(X; L) = 0$ for all $k > 0$.

Proposition 12 *Let $0 \to A \xrightarrow{d^\bullet} L^\bullet$ be a resolution of A by acyclic sheaves. Then there is a natural isomorphism*

$$\gamma : H^k(\Gamma(X; L^\bullet)) \to H^k(X; A).$$

This means that sheaf cohomology may be computed using acyclic resolutions instead of injective ones.

Proof We split the resolution into short exact sequences

$$0 \to M^k \to L^k \to M^{k+1} \to 0$$

where $M^k = \ker(d^k) = \mathrm{Im}(d^{k-1})$; in particular $M^0 = A$. The induced long exact sequence in this case takes the form

$$\cdots \to H^p(X; M^k) \longrightarrow H^p(X; L^k) \longrightarrow H^p(X; M^{k+1}) \longrightarrow$$
$$\longrightarrow H^{p+1}(X; M^k) \longrightarrow H^{p+1}(X; L^k) \to \cdots$$

Since $L^\bullet$ is acyclic, we find isomorphisms $H^p(X; M^{k+1}) \cong H^{p+1}(X; M^k)$ for each $p > 0$.

When $p = 0$ we have the following segment of the sequence

$$0 \to \Gamma(X; M^k) \longrightarrow \Gamma(X; L^k) \longrightarrow \Gamma(X; M^{k+1}) \longrightarrow H^1(X; M^k) \to 0$$

which gives an isomorphism

$$H^{k+1}(\Gamma(X; L^\bullet)) = \frac{\Gamma(X; M^{k+1})}{\mathrm{Im}(\Gamma(X; L^k) \to \Gamma(X; M^{k+1}))} \cong H^1(X; M^k)$$

Combining the previous isomorphisms we get

$$H^{k+1}(\Gamma(X; L^\bullet)) \cong H^1(X; M^k) \cong H^2(X; M^{k-1}) \cong \cdots$$
$$\cdots \cong H^{k+1}(X; M^0) = H^{k+1}(X; A)$$

□

The following types of sheaves are examples of acyclic sheaves. The reader may consult the proofs in Theorems 5.5, 9.11 and 9.16 of [3].

Definition 40 Let A be a sheaf over X.

- A is flabby if for every open $U \subset X$, the restriction map $\Gamma(X; A) \to \Gamma(U; A)$ is surjective.
- A is soft if for every closed $Z \subset X$, the restriction map $\Gamma(X; A) \to \Gamma(Z; A)$ is surjective.
- If A is a sheaf of O_X-modules, A is called fine if for every open cover $\{U_i\}$ of X there is a partition of unity, i.e., a family of sections $s_i \in \Gamma(X; O_X)$ such that $\mathrm{supp}(s_i) \subset U_i$ and $\sum_i s_i = 1$. The sum is assumed to be locally finite, this means that for every $x \in X$ there is a neighbourhood where all but finite of the sections vanish.

Example 18 Let X be a smooth manifold. For every open $U \subset X$ let $C^\infty(U)$ denote the sheaf of smooth functions on U with real values. C^∞ is easily seen to be a sheaf of rings over X, hence (X, C^∞) is a ringed space. We know that the spaces of differential forms $\Omega^k(U)$ are modules over $C^\infty(U)$, so Ω^k can be seen to be a sheaf of C^∞-modules. The existence of partitions of unity over X guarantees that the sheaves Ω^k are fine sheaves.

Consider now the constant sheaf $\mathbb{R}_X$ that assigns the ring $\mathbb{R}$ to every non-empty open set of X with restrictions given by the identity. The inclusion $\mathbb{R} \hookrightarrow C^\infty(U) = \Omega^0(U)$

as constant functions provides a map of sheaves $\mathbb{R}_X \to \Omega^0$. Then the de Rham differential gives us a sequence

$$0 \to \mathbb{R}_X \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$$

For every $x \in X$ we have a neighbourhood U diffeomorphic to $\mathbb{R}^n$, by Poincaré's Lemma we have that

$$0 \to \mathbb{R} \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots$$

is exact. Then the sequence is exact at the level of sheaves and $0 \to \mathbb{R}_X \xrightarrow{d} \Omega^\bullet$ is an acyclic resolution for the constant sheaf. The cohomology of X with values in the constant sheaf $\mathbb{R}_X$ is just the de Rham cohomology of X:

$$H^\bullet(X; \mathbb{R}_X) = H^\bullet_{\mathrm{DR}}(X).$$

Finally, we define constructible sheaves.

Definition 41 A is said to be constructible if there is a partition $X = \bigsqcup X_i$ where the X_i are locally closed subsets and $A|_{X_i}$ is a locally constant sheaf for every i.

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Homogeneous Spaces of Operators

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Abstract Let $\mathcal{A}$ be a unital C^*-algebra. In this chapter (and in the next one) we study the Poincaré half-space $\mathcal{H}$ of $\mathcal{A}$

$$\mathcal{H} := \{h \in \mathcal{A} : Im(h) \text{ is positive and invertible}\}.$$

In the case of the C^*-algebra $\mathcal{B}(\mathcal{L})$ of bounded linear operators on the Hilbert space $\mathcal{L}$, the space $\mathcal{H}$ is isomorphic to the tangent bundle of the set of inner products on $\mathcal{L}$. Also consider the following situation. Let E be a complex vector bundle over a compact space M provided with a Hermitian structure. Consider the C^*-algebra $\Gamma(\mathrm{End}(E))$ of continuous sections of the endomorphism bundle of E. The Poincaré space of $\Gamma(\mathrm{End}(E))$ consists of the elements of the form $X + ia$, where a is a Hermitian structure on E and X is infinitesimal deformation of a. The space $\mathcal{H}$ is naturally isomorphic to the following spaces: 1. the Poincaré disk $\mathcal{D}$ of $\mathcal{A}$:

$$\mathcal{D} := \{z \in \mathcal{A} : \|z\| < 1\};$$

2. the space $\mathcal{Q}_\rho$ of $\mathcal{A}$-module projections q acting in $\mathcal{A} \times \mathcal{A}$ which decompose the form $\theta = \langle \rho\, \cdot, \cdot \rangle$ induced by a fixed symmetry ρ in $\mathcal{A} \times \mathcal{A}$, in the following sense:

$$\theta \text{ is positive in } R(q) \text{ and negative in } N(q).$$

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These spaces are homogeneous reductive spaces of the groups $\mathcal{U}_\rho$ and $\mathcal{U}_{\rho'}$ of invertible 2×2 matrices with entries in $\mathcal{A}$ which leave an appropriate form invariant. The isomorphisms between these spaces are equivariant with respect to these actions, and the actions are isometric. We regard these spaces as alternative presentations of $\mathcal{H}$, each one having its particular features which allow us to underline different aspects of the geometry of $\mathcal{H}$. For instance, from $\mathcal{Q}_\rho$, $\mathcal{H}$ inherits the hyperbolic structure (it behaves as a manifold of non positive curvature).

1 Introduction

In this work, we study the Poincaré half-space of a unital C*-algebra $\mathcal{A}$,

$$\mathcal{H} = \{h \in \mathcal{A} : Im\ h = \frac{1}{2i}(h - h^*) \text{ is a positive invertible element of } \mathcal{A}\}.$$

We consider $\mathcal{H}$ as a generalization of the classical Poincaré half-plane,

$$\{h \in \mathbb{C} : Im\ h > 0\}.$$

First, we introduce a natural group of movements of $\mathcal{H}$, which we denote $\mathcal{U}_{\rho'}$. This is a subgroup of G_{M_2}, the group of invertible 2×2 matrices with entries in $\mathcal{A}$; more precisely, $\mathcal{U}_{\rho'}$ is the unitary group of the form

$$\theta_{\rho'} : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}\,, \quad \theta_{\rho'}(\mathbf{x}, \mathbf{y}) = -i(x_1^* y_2 - x_2^* y_1),$$

for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{A}^2$.

One of the motivations of this study is the following. Denote

$$G_{\mathcal{A}} = \{c \in \mathcal{A} : c \text{ is invertible}\}$$

and

$$G^+ = \{a \in G_{\mathcal{A}} : a \text{ is positive}\}.$$

We consider G^+ as the configuration space of a quantum mechanical system whose elements represent a family of equivalent metrics over a Hilbert space. The tangent bundle TG^+ is the phase space of the configuration space G^+ of the quantum system. Note also that TG^+ identifies in a natural way with the bundle of observables associated with the different metrics. We shall explain this in more detail below.

Note the correspondence

$$(a, X) \longleftrightarrow X + ia,$$

where $a \in G^+$ and $X \in (TG^+)_a$; here X is a selfadjoint element of $\mathcal{A}$, because G^+ is open in the space of selfadjoint elements of $\mathcal{A}$. This correspondence establishes a clear identification between TG^+ and $\mathcal{H}$. Following ideas of Siegel [1], we define a transitive action of $\mathcal{U}_{\rho'}$ on $\mathcal{H}$, which makes $\mathcal{H}$ into a homogeneous space of $\mathcal{U}_{\rho'}$. Therefore, TG^+ can be considered as the phase space of the mentioned quantum system and $\mathcal{U}_{\rho'}$ as its group of symmetries.

$\mathcal{U}_{\rho'}$ is a generalization of the group of Möbius transformations of the Poincaré half-plane.

It should be mentioned that there exist, as in the classical case, an identification of $\mathcal{H}$ with the Poincaré disk

$$\mathcal{D} = \{z \in \mathcal{A} : \|z\| < 1\}.$$

$\mathcal{D}$ is a homogeneous space of the group $\mathcal{U}_\rho \subset G_{M_2}$ which are unitary with respect to the form

$$\theta_\rho : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}\,, \quad \theta_\rho(\mathbf{x}, \mathbf{y}) = x_1^* y_1 - x_2^* y_2.$$

The Möbius transformation associated to the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

is an isomorphism of homogeneous spaces between $\mathcal{D}$ and $\mathcal{H}$. With this identification, we can use $\mathcal{D}$ or $\mathcal{H}$ according to the computational convenience.

$\mathcal{H}$ is a homogeneous reductive space of $\mathcal{U}_{\rho'}$, and as such it has a canonical connection which produces parallelism, geodesics, etc. On $\mathcal{H}$ several Finsler metrics can be defined, which are invariant under the action of $\mathcal{U}_{\rho'}$. For one of these metrics. there is an isometric immersion of $\mathcal{H}$ in the space $G^+_{M_2}$ of positive invertible elements in M_2. Therefore, $\mathcal{H}$ inherits the metric properties of $G^+_{M_2}$. For instance, the geodesics of $\mathcal{H}$ are the shortest curves joining its endpoints, and $\mathcal{H}$ becomes a non-positive curvature space in the sense of Alexandroff [2–4].

The space $\mathcal{H}$ admits a non-commutative Kähler-type structure which is equivariant under the action of $\mathcal{U}_{\rho'}$. More precisely, $\mathcal{H}$ carries a complex structure and a Hermitian product whose imaginary part is a symplectic form, which is exact, in the sense that it is the curvature of a connection on a vector bundle on $\mathcal{H}$. We define a fiber bundle C in C*-algebras, which we call the coefficient bundle of $\mathcal{H}$ (or $\mathcal{D}$), whose fibres are isomorphic to $\mathcal{A}$. It is on this coefficient bundle that the Hermitian product, the symplectic form and so on, take their values.

If $\mathcal{A}$ has a trace, the traces of these forms are classical scalar-valued (Hilbertian, symplectic) forms.

We construct on $\mathcal{H}$ a moment map with values in the dual of the Banach-Lie algebra of $\mathcal{U}_{\rho'}$, and we prove the convexity of its image (a kind of Atiyah–Guillemin–Sternberg result [5, 6]).

In the last part of this monograph we relate the metric properties of $\mathcal{H}$ as a subset of the "projective line" of $\mathcal{A}$. In order to describe the most important result we need to use the model $\mathcal{D}$. Given $z_0, z_1 \in \mathcal{D}$, let $\delta : \mathbb{R} \to \mathcal{D}$ be the unique geodesic with $\delta(0) = z_0$ and $\delta(1) = z_1$, and it limit points

$$z_{+\infty} = \lim_{t \to +\infty} \delta(t) \ , \quad z_{-\infty} = \lim_{t \to -\infty} \delta(t).$$

Using ideas of Zelikin [7], we define the cross ratio of the ordered set $\{z_{-\infty}, z_0, z_1, z_{+\infty}\}$, which is an operator $cr(z_{-\infty}, z_0, z_1, z_{+\infty})$. On the other hand, we have the "initial velocity operator" $\dot{\delta}(0)$, whose norm is the distance between z_0 and z_1. We establish an identity which relates the distance operator $\dot{\delta}(0)$ and the cross ratio operator $cr(z_{-\infty}, z_0, z_1, z_{+\infty})$. This identity is an operator valued version of the classical formula involving the hyperbolic distance in the complex unit disk with the cross ratio in the complex projective line.

The reader will find that we study $\mathcal{H}$ and $\mathcal{D}$ as reductive homogeneous spaces of $\mathcal{U}_{\rho'}$ and $\mathcal{U}_{\rho}$, respectively, by using different models for these spaces. Namely, spaces of idempotents in M_2 which we call $\mathcal{Q}_{\rho'}$ and $\mathcal{Q}_{\rho}$, with a certain relation to ρ' and ρ. One of the advantages of this approach is that $\mathcal{Q}_{\rho'}$ and $\mathcal{Q}_{\rho}$ are subsets of the manifold $\mathcal{Q}$ of all idempotents in M_2, a space whose differential geometry is well understood. The group $\mathcal{U}_{\rho}$ acts on $\mathcal{Q}_{\rho}$ by similarity: $a \cdot q = aqa^{-1}$, for $a \in \mathcal{U}_{\rho}$ and $q \in \mathcal{Q}_{\rho}$. Thus, for $\mathcal{Q}_{\rho'}$ and $\mathcal{Q}_{\rho}$, all computations are done in M_2. It should be mentioned that $\mathcal{Q}_{\rho'}$ and $\mathcal{Q}_{\rho}$ are conjugated by the unitary matrix U defined above.

The contents of this chapter and the next one are the following.

In Sect. 2 we begin by describing the differential geometry of G^+. We show how the polar decomposition allows one to regard sets of idempotents as subsets of G^+. Given a symmetry (a selfadjoint unitary element) the unitary group of the quadratic form induced by the symmetry is studied. Our focus is afterwards on the following symmetries in the algebra M_2:

$$\rho' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We introduce the main objects of our study: the (equivalent) spaces $\mathcal{Q}_{\rho}$ and $\mathcal{Q}_{\rho'}$, $\mathcal{D}$ and $\mathcal{H}$, with their structure as homogeneous spaces of the unitary groups $\mathcal{U}_{\rho'}$ or $\mathcal{U}_{\rho}$. Auxiliary homogeneous spaces are also introduced, which will play a central role, the hyperboloids $\mathcal{K}_{\rho'}$ and $\mathcal{K}_{\rho}$, with the maps and diagrams which relate them to the aforementioned homogeneous spaces.

Section 3 is devoted to examine the metric properties of $\mathcal{Q}_{\rho}$, $\mathcal{D}$ and $\mathcal{H}$.

In Sect. 1 of Chap. 4 we introduce several vector bundles which allow us to lay the basic formal tools used in the later sections. Among them, the coefficient bundle $\mathcal{C}$, a vector bundle whose fibers are C*-algebras isomorphic to $\mathcal{A}$.

In Sect. 2 of Chap. 4 we introduce the complex structure of $\mathcal{Q}_\rho$ (and of $\mathcal{D}$ and $\mathcal{H}$). A Hilbertian inner product is defined, which takes values in the coefficient bundle C. Here we show that the symplectic form ω (also C-valued) induced by this inner product is exact: it is the curvature of the canonical connection of $\mathcal{Q}_\rho$.

In Sect. 3 of Chap. 4 we produce the results described above, concerning the projective nature of the space $\mathcal{Q}_\rho$.

2 Homogeneous Spaces

2.1 The Geometry of G^+

Since one of the motivations of this work is the identification of $\mathcal{H}$ with the tangent bundle TG^+, we begin with a summary of results on the geometry of G^+. Moreover, the geometry of $\mathcal{D}$ and $\mathcal{H}$ depend on the geometry of G^+. The first study of G^+ that we are aware of, can be found in [8]. The results of this section can be found in [9–14]. For other examples of homogeneous spaces of groups of invertible elements in C*-algebras see [15–26].

First we fix notation for this section. $\mathcal{A}$ is a unital C*-algebra, $\mathcal{A}_h = \{x \in \mathcal{A} : x^* = x\}$ (Hermitian elements of $\mathcal{A}$), $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$ (anti-Hermitian elements of $\mathcal{A}$); recall that $\mathcal{A}_h \oplus \mathcal{A}_{ah} = \mathcal{A}$. $G_\mathcal{A}$ is the group of invertible elements of $\mathcal{A}$, G^+ is the space of positive elements of $G_\mathcal{A}$, $\mathcal{U}_\mathcal{A}$ is the group of unitary elements of $\mathcal{A}$.

If $\mathcal{A}$ is represented in a Hilbert space $(\mathcal{L}, \langle\ ,\ \rangle)$, then G^+ can be thought of as a set of inner product on $\mathcal{L}$ which are equivalent to $\langle\ ,\ \rangle$: if $a \in G^+$,

$$\langle \xi, \eta \rangle_a := \langle a\xi, \eta \rangle,\ \ \xi, \eta \in \mathcal{L}.$$

Each $a \in G^+$ defines also an involution ^{*_a} in $\mathcal{A}$: for $x \in \mathcal{A}$, $x^{*_a} = a^{-1}xa$. If $\mathcal{A}$ is represented in $\mathcal{L}$, then ^{*_a} is the adjoint with respect to $\langle\ ,\ \rangle_a$.

Note that for $a \in G^+$, if we denote

$$\mathcal{A}_h^a := \{x \in \mathcal{A} : x^{*_a} = x\} \quad \text{and} \quad \mathcal{A}_{ah}^a := \{x \in \mathcal{A} : x^{*_a} = -x\},$$

then it holds that $\mathcal{A} = \mathcal{A}_h^a \oplus \mathcal{A}_{ah}^a$ (i.e., the decomposition as a-Hermitian and a-anti-Hermitian elements).

Since $G_\mathcal{A}$ is an open subset of $\mathcal{A}$, its manifold structure is trivial, and for each $g \in G_\mathcal{A}$, the tangent space $(TG_\mathcal{A})_g$ identifies with $\mathcal{A}$. Similarly, G^+ is open in $\mathcal{A}_h$, and thus $(TG^+)_a \simeq \mathcal{A}_h$ for each $a \in G^+$.

The group $G_\mathcal{A}$ acts on G^+ by means of

$$L : G_\mathcal{A} \times G^+ \to G^+\ ,\ \ L_g a = (g^*)^{-1} a g^{-1},$$

for $g \in G_{\mathcal{A}}$ and $a \in G^+$. The action is transitive: for each pair $a, b \in G^+$ it holds that $L_g a = b$ for $g = b^{-1/2}a^{1/2}$.

Along this paper, for the left action of a group G on a manifold M, we use both notations $L_g m$ and $g \cdot m$, for $g \in G$ and $m \in M$.

The *isotropy subgroup* of $a \in G^+$ for the action L is

$$\mathcal{I}_a = \{g \in G_{\mathcal{A}} : (g^*)^{-1} a g^{-1} = a\}.$$

Note that $\mathcal{I}_1 = \mathcal{U}_{\mathcal{A}}$. Again, if $\mathcal{A}$ is represented in $\mathcal{L}$, then

$$\mathcal{I}_a = \{g \in G_{\mathcal{A}} : g : (\mathcal{L}, \langle\ ,\ \rangle_a) \to (\mathcal{L}, \langle\ ,\ \rangle_a) \text{ is unitary}\}.$$

For any $a \in G^+$, the map

$$\mathbf{p}_a : G_{\mathcal{A}} \to G^+,\ \mathbf{p}_a(g) = L_g a = (g^*)^{-1} a g^{-1}$$

is a submersion with a global section $\mathbf{s}_a : G^+ \to G_{\mathcal{A}}$, $\mathbf{s}_a(b) = b^{-1/2}a^{1/2}$. As a consequence, we get a diffeomorphism

$$G_{\mathcal{A}}/\mathcal{U}_{\mathcal{A}} \longleftrightarrow G^+,\ g\mathcal{U}_{\mathcal{A}} \mapsto (g^*)^{-1} g^{-1} = (gg^*)^{-1},$$

whose inverse is $b \mapsto b^{-1/2}\mathcal{U}_{\mathcal{A}}$.

Notice that the tangent map of $\mathbf{p}_a$ at $1 \in G_{\mathcal{A}}$ is

$$(T\mathbf{p}_a)_1 : \mathcal{A} \to \mathcal{A}_h,\ T(\mathbf{p}_a)_1 x = -(ax + x^* a).$$

Its kernel is the Banach-Lie algebra of $\mathcal{I}_a$:

$$\mathfrak{i}_a = N((T\mathbf{p}_a)_1) = (T\mathcal{I}_a)_1.$$

If $\mathcal{A}$ is represented in $\mathcal{L}$, then $(T\mathcal{I}_a)_1$ consists of the elements of $\mathcal{A}$ which are anti-Hermitian operators in $(\mathcal{L}, \langle\ ,\ \rangle_a)$. Therefore, a natural complement for $(T\mathcal{I}_a)_1$ in $(TG_{\mathcal{A}})_1 = \mathcal{A}$ is

$$\mathbf{H}_a = \{x \in \mathcal{A} : ax = x^* a\},$$

i.e., the a-Hermitian elements of $\mathcal{A}$. It is a relevant fact that the restriction

$$(T\mathbf{p}_a)_1|_{\mathbf{H}_a} : \mathbf{H}_a \to \mathcal{A}_h$$

is a linear isomorphism, whose inverse is

$$\mathbf{K}_a : \mathcal{A}_h \to \mathbf{H}_a,\ \mathbf{K}_a(z) = -\frac{1}{2} a^{-1} z.$$

It is easy to verify that $\mathbf{H}_a$ has the following properties:

1. for $g \in \mathcal{I}_a$ and $y \in \mathbf{H}_a$, it holds that $gyg^{-1} \in \mathbf{H}_a$;
2. for $x \in \mathfrak{i}_a = (T\mathcal{I}_a)_1$ and $y \in \mathbf{H}_a$, it holds that $[x, y] = xy - yx \in \mathbf{H}_a$;
3. the map $a \mapsto \mathbf{H}_a$ is smooth.

This shows that the distribution of subspaces $a \mapsto \mathbf{H}_a$ defines a *reductive structure* in G^+. Usually, $\mathbf{H}_a$ is called the *horizontal space* of $\mathcal{A} = (TG_{\mathcal{A}})_a$ at $a \in G^+$.

The map $a \mapsto \mathbf{K}_a$ will be called the 1-*form* of the reductive homogeneous space G^+.

As with any reductive homogenous space, for any smooth curve $a = a_t : [0, 1] \to G^+$ there exists a unique *horizontal lifting* of a, i.e., a smooth curve $g = g_t : [0, 1] \to G$ such that $\mathbf{p}_{a_0} g_{=} a$ and $\dot{g}_t := \frac{dg}{dt} \in \mathbf{H}_{g_t}$ for all $t \in [0, 1]$. The horizontal lifting g is the unique solution of the linear differential equation

$$\begin{cases} \dot{g} = \mathbf{K}_a(\dot{a})g \\ g_0 = 1, \end{cases}$$

which we call the *transport equation* for G^+. The solution g provides the parallel transport of vectors along the curve a. Explicitly, the transport equation for G^+ is

$$\begin{cases} \dot{g} = -\frac{1}{2}a^{-1}\dot{a}g \\ g_0 = 1. \end{cases}$$

The solution g is used to define (via parallel transport) the *covariant derivative* of a vector field $X = X_t$ along $a = a_t$:

$$\frac{DX}{dt} := (g_t^*)^{-1}\left(\frac{d}{dt}(TL_{g_t})_{a_t} X_t\right) g_t^{-1} = \dot{X} - \frac{1}{2}(Xa^{-1}\dot{a} + \dot{a}a^{-1}X).$$

The vector field is *parallel* if $\dfrac{DX}{dt} = 0$. A curve $a = a_t$ is called a *geodesic* of this linear connection if $\dot{a}$ is parallel, i.e.,

$$\ddot{a} = \dot{a}a^{-1}\dot{a}. \tag{1}$$

There are two classical problems associated to the Eq. (1), namely:

1. the *initial values* problem

$$\begin{cases} \ddot{a} = \dot{a}a^{-1}\dot{a} \\ a_0 = b \\ \dot{a}_0 = x, \quad \text{for } b \in G^+, x \in (TG^+)_b \end{cases}$$

and

2. the *boundary values* problem

$$\begin{cases} \ddot{a} = \dot{a}a^{-1}\dot{a} \\ a_0 = b \\ a_1 = c, \quad \text{for } b, c \in G^+ \end{cases}$$

Both have a unique solution. For the initial values problem the solution is

$$a_t = e^{-\frac{t}{2}x} = e^{t\mathbf{K}_1(x)} \text{ if } b = 1,$$

or

$$a_t = b^{1/2} e^{-\frac{t}{2} b^{-1/2} x b^{-1/2}} b^{1/2} \tag{2}$$

for $b \in G^+$ and $x \in \mathcal{A}_h = (TG^+)_1$ arbitrary. The solution for the boundary values problem is

$$a_t = b^{1/2}(b^{-1/2} c b^{-1/2})^t b^{1/2}. \tag{3}$$

Note that the velocity vector at $t = 0$ of this geodesic is $x = -2\log(b^{-1/2} c b^{-1/2})$

The *exponential map* of G^+ is therefore given by

$$\exp_a : (TG^+)_a \simeq \mathcal{A}_h \to G^+ , \quad \exp_a(x) = a_1 = a^{1/2} e^{a^{-1/2} x a^{-1/2}} a^{1/2}, \tag{4}$$

which is a global diffeomorphism with inverse

$$\log_a : G^+ \to \mathcal{A}_h , \quad \log_a b = a^{1/2} \log(a^{-1/2} b a^{-1/2}) a^{1/2}. \tag{5}$$

One also has a natural metric in G^+. We define a distance d in G^+ which is invariant under the action of $G_{\mathcal{A}}$:

$$d(L_g a, L_g b) = d(a, b) \text{ for all } a, b \in G^+, g \in G_{\mathcal{A}}. \tag{6}$$

To obtain this metric, we define a continuous distribution $G^+ \ni a \mapsto \| \ \|_a$ of norms on the tangent spaces $(TG^+)_a \simeq \mathcal{A}$. This is usually called a *Finsler metric* (though no smoothness nor nondegeneracy requirements are made, as in classic finite dimensional Finsler geometry [27]). In order that the metric d in G^+ induced by this distribution of norms in the tangent bundle satisfies (6), we need that the distribution verifies

$$\|x\|_b = \|L_{g^{-1}} x\|_a = \|g^* x g\|_a$$

for any $x \in \mathcal{A}$ and any $a, b \in G^+$, $g \in G_{\mathcal{A}}$ such that $L_g a = b$.

We define:

1. $\|x\|_1 := \|x\|$, the usual norm in $\mathcal{A}$, which imposes us that
2. $\|x\|_a := \|a^{-1/2} x a^{-1/2}\|$.

It is clear that this distribution of norms is invariant under the action of $G_{\mathcal{A}}$, by construction.

Definition 1 For $b, c \in G^+$, the (Finsler) distance $d(b, c)$ is defined by

$$d(b, c) = \inf\{length(\gamma) : \gamma \text{ is smooth}, \gamma : b \sim c\}.$$

Theorem 1 *1. If $b, c \in G^+$ and $a_t = b^{1/2}(b^{-1/2}cb^{-1/2})^t b^{1/2}$ is the unique geodesic joining b and c, then $a = a_t$ is shortest among all smooth curves joining b and c in G^+. Then*

$$d(b, c) = length(a) = \|\log(b^{-1/2}cb^{-1/2})\|.$$

2. If $a = a_t$, $b = b_t$ are geodesics in G^+ such that $a_0 = b_0$, then

$$d(a_t, b_t) \leq t d(a_1, b_1) \tag{7}$$

for all $t \in [0, 1]$.

3. If $a = a_t$, $b = b_t$ are geodesics in G^+, then $f(t) = d(a_t, b_t)$ is a convex function in $\mathbb{R}$

Remark 1 1. The distance d is $G_{\mathcal{A}}$-invariant.
2. Note that the geodesics are defined for all $t \in \mathbb{R}$.
3. Metric spaces with these properties are called *non positive curvature spaces* in the sense of Alexandroff [2–4].
4. In a family of cones in Banach spaces there is a distance named *Thompson distance*, defined by

$$d_T(a, b) = \log \max\{\inf\{r > 0 : a \leq rb\}, \inf\{s > 0 : b \leq sa\}\}.$$

It turns out that d_T coincides with the distance just defined in the cone G^+.

5. Suppose that $(X, \|\ \|)$ is a Banach space and denote by $\mathcal{X}$ the set of all norms in X which are equivalent to $\|\ \|$. The *Banach–Mazur distance* on $\mathcal{X}$ is defined by

$$d_{BM}(\|\ \|_1, \|\ \|_2) = \max\{\log \sup_{\xi \neq 0} \frac{\|\xi\|_1}{\|\xi\|_2}, \log \sup_{\xi \neq 0} \frac{\|\xi\|_2}{\|\xi\|_1}\}.$$

If $(\mathcal{L}, \|\ \|)$ is a Hilbert space, then $\|\xi\|_a := \langle a\xi, \xi\rangle^{1/2} = \|a^{1/2}\xi\|$ for $a \in G^+$ is a norm equivalent to $\|\ \| = \langle\ ,\ \rangle^{1/2}$. Then

$$2d_{BM}(\|\ \|_a, \|\ \|_b) = d(a, b).$$

6. In the above context (of equivalent norms in $\mathcal{L}$), if one applies the complex interpolation method of Calderón–Lions to the norms $\|\ \|_a$ and $\|\ \|_b$ $(a, b \in G^+)$, at time $t \in [0, 1]$ one obtains the norm $\|\ \|_{a_t}$ induced by the geodesic a_t of G^+ joining $a_0 = a$ and $a_1 = b$.

2.2 Polar Decomposition of Reflections

If T is a bounded linear operator on a Hilbert space, there exists a unique partial isometry V such that $T = V|T| = |T^*|V$, with $|T| = (T^*T)^{1/2}$ such that $N(V) = N(T)$. If T is invertible, V turns out to be unitary. In an abstract C*-algebra $\mathcal{A}$ a similar result holds, except that the partial isometry may not belong to $\mathcal{A}$ but to $\mathcal{A}^{**}$, the enveloping von Neumann algebra of $\mathcal{A}$. The result is known as the *polar decomposition* of a bounded linear operator on a Hilbert space. Let us state it for further use:

Theorem 2 *For every bounded linear operator T on a Hilbert space $\mathcal{E}$, there exists a unique partial isometry V such that*

$$T = V|T| = |T^*|V$$

and $N(V) = N(T)$, where $|T| = (T^*T)^{1/2}$.

This decomposition is also available for invertible elements of a (unital) C*-algebra $\mathcal{A}$, in which case the isometric part is a unitary element of $\mathcal{A}$. More precisely:

Proposition 1 *Let $a \in \mathcal{A}$ be invertible. Then*

1. $\sigma_a = a|a|^{-1} = |a^*|^{-1}a \in \mathcal{U}_{\mathcal{A}}$.
2. *If $u, v \in \mathcal{U}_{\mathcal{A}}$, $\lambda, \mu \in G^+$ and $a = u\lambda = \mu v$, then $u = v = \sigma_a$, $\lambda = |a|$ and $\mu = |a^*|$.*

Proof First note that $a(a^*a)^{1/2} = (aa^*)^{1/2}a$, a fact which follows from functional calculus. It is easy to verify that σ_a is a unitary element. If $a = u\lambda$ then $\sigma_a|a| = u\lambda$ and $\lambda|a|^{-1} = u^*\sigma_a \in \mathcal{U}_{\mathcal{A}}$. Note that the spectrum $sp(\lambda|a|) \subset (0, +\infty)$. Indeed, recall that if x, y are invertible elements in a Banach algebra, then $sp(xy) = sp(yx)$; thus, $sp(\lambda|a|) = sp(\lambda|a|^{1/2}|a|^{1/2}) = sp(|a|^{1/2}\lambda|a|^{1/2}) \subset (0, +\infty)$, because $|a|^{1/2}\lambda|a|^{1/2} \in G^+$. Then $\lambda|a|^{-1} = u^*\sigma_a = 1$, because $1 \in \mathcal{A}$ is the unique unitary element with positive spectrum. Similarly, $\mu = |a^*|$ and $v = \sigma_a$. □

Remark 2 It holds that

$$a^2 = \sigma_a|a||a^*|\sigma_a = |a^*|\sigma_a^2|a| \tag{8}$$

Corollary 1 *For $a \in G_{\mathcal{A}}$, the following conditions are equivalent:*

1. $|a||a^*| = 1$.
2. $\sigma_a|a| = |a|^{-1}\sigma_a$.
3. $a^2 = \sigma_a^2$.
4. $a^2 \in \mathcal{U}_{\mathcal{A}}$.

Corollary 2 *If $a \in G_{\mathcal{A}}$ then $\sigma_a^2 = 1$ if and only if $a^2 = |a^*||a|$.*

Corollary 3 *For $a \in G_{\mathcal{A}}$, the following conditions are equivalent:*

1. $a^2 = 1$.
2. $a^2 \in \mathcal{U}_{\mathcal{A}}$ *and* $\sigma_a^2 = 1$.
3. $\sigma_a^2 = 1$ *and* $\sigma_a|a| = |a|^{-1}\sigma_a$.
4. $\sigma_a^2 = 1$ *and* $|a^*||a| = 1$.
5. $\sigma_a^2 = a^2 = |a^*||a|$.

The results of this section can be found, in a slightly different form, in [9, 28].

2.3 A Space of Idempotents Related to a Symmetry

From now on, we shall deal with the C*-algebra $M_2 = M_2(\mathcal{A}) = \mathcal{A}^{2\times 2}$, which can be regarded as the algebra of all $\mathcal{A}$-linear adjointable operators acting in the right C* $\mathcal{A}$-module $\mathcal{A}^2$. The $\mathcal{A}$-valued inner product of this module is given by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^*\mathbf{y} = x_1^* y_1 + x_2^* y_2,$$

for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. The adjoint of a matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2$ is given by

$$a^* = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}.$$

If $a \in M_2$ and $\mathcal{S}$ is a submodule of $\mathcal{A}^2$, we write $a \geq 0$ in $\mathcal{S}$ if $\langle a\mathbf{x}, \mathbf{x}\rangle \geq 0$ for all $\mathbf{x} \in \mathcal{S}$.

General references on C* $\mathcal{A}$-modules are [29, 30].

We define

$$\mathcal{Q} = \{q \in M_2 : q^2 = q\} \tag{9}$$

and

$$\mathcal{P} = \{p \in \mathcal{Q} : p^* = p\}. \tag{10}$$

Notice that the map $q \mapsto 2q - 1$ establishes natural bijections

$$\mathcal{Q} \to \{a \in M_2 : a^2 = 1\}$$

and

$$\mathcal{P} \to \{\sigma \in M_2 : \sigma = \sigma^{-1} = \sigma^*\}.$$

Any $a \in \mathcal{A}$ such that $a^2 = 1$ is called a *reflection*, and selfadjoint (i.e. unitary) reflections are called *symmetries*. Note that σ is a symmetry if and only if $\frac{1}{2}(1 + \sigma) \in \mathcal{P}$.

Proposition 2 *Given $q \in Q$ and a symmetry σ, the following conditions are equivalent:*

1. $\sigma(2q-1) \in G_2^+$.
2. *There exists* $\lambda \in G_2^+$ *such that* $2q - 1 = \lambda^2\sigma$.
3. *There exists* $\lambda \in G_2^+$ *such that* $q = \lambda(\frac{1}{2}(1+\sigma))\lambda^{-1}$ *and* $\sigma\lambda\sigma = \lambda^{-1}$.
4. $\sigma \geq 0$ *in* $R(q)$, $\sigma \leq 0$ *in* $N(q)$ *and* $q = \sigma q^*\sigma$.

Proof Note that

$$\mathcal{A}^2 = R(q)\dot{+}N(q) = R(q)\dot{+}R(1-q),$$

so every $\mathbf{x} \in \mathcal{A}^2$ can be written $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$ with $\mathbf{x}_0 = q\mathbf{x}_0$ and $\mathbf{x}_1 = (1-q)\mathbf{x}_1$.

1. $\Longrightarrow$ 2. If $\sigma(2q-1) \in G_2^+$, then the polar decomposition of $2q-1$ is

$$2q - 1 = \sigma(\sigma(2q-1)) = (\sigma(2q-1))^{-1}\sigma = ((2q-1)\sigma)\sigma.$$

Take $\lambda = ((2q-1)\sigma)^{1/2}$.

2. $\Longrightarrow$ 3. From $2q - 1 = \lambda^2\sigma$, we get $q = \lambda(\frac{1}{2}(1+\sigma))\lambda^{-1}$. The identity $\sigma\lambda\sigma = \lambda^{-1}$ follows from Corollary 3, part 3.

3. $\Longrightarrow$ 1. $\sigma(2q-1) = ((2q-1)\sigma)^* = (\lambda^2)^* = \lambda^2 > 0$. Thus 1., 2. and 3. are equivalent.

4. $\Longrightarrow$ 1. From the remark above, if $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$, we get $(2q-1)\mathbf{x} = \mathbf{x}_0 - \mathbf{x}_1$. Then

$$\langle\sigma(2q-1)\mathbf{x}, \mathbf{x}\rangle = \langle\sigma\mathbf{x}_0 - \sigma\mathbf{x}_1, \mathbf{x}_0 + \mathbf{x}_1\rangle = \langle\sigma\mathbf{x}_0, \mathbf{x}_0\rangle - \langle\sigma\mathbf{x}_1, \mathbf{x}_1\rangle + \langle\sigma\mathbf{x}_0, \mathbf{x}_1\rangle - \langle\sigma\mathbf{x}_1, \mathbf{x}_0\rangle.$$

Note that

$$\langle\sigma\mathbf{x}_0, \mathbf{x}_1\rangle = \langle\sigma q\mathbf{x}_0, \mathbf{x}_1\rangle = \langle q^*\sigma\mathbf{x}_0, \mathbf{x}_1\rangle = \langle\sigma\mathbf{x}_1, q\mathbf{x}_1\rangle = 0$$

and

$$\langle\sigma\mathbf{x}_1, \mathbf{x}_0\rangle = \langle\mathbf{x}_1, \sigma\mathbf{x}_0\rangle = \langle\sigma\mathbf{x}_0, \mathbf{x}_1\rangle^* = 0.$$

On the other hand, by 4.

$$\langle\sigma\mathbf{x}_0, \mathbf{x}_0\rangle \geq 0 \quad \text{and} \quad \langle\sigma\mathbf{x}_1, \mathbf{x}_1\rangle \leq 0.$$

Thus, $\sigma(2q-1) \geq 0$ and clearly $\sigma(2q-1) \in G_{M_2}$.

2. $\Longrightarrow$ 4. By 2. $\sigma = \lambda^{-2}(2q-1)$. If $\mathbf{x}_0 = q\mathbf{x}_0$,

$$\langle\sigma\mathbf{x}_0, \mathbf{x}_0\rangle = \langle\lambda^{-2}(2q-1)\mathbf{x}_0, \mathbf{x}_0\rangle = \langle\lambda^{-2}\mathbf{x}_0, \mathbf{x}_0\rangle \geq 0.$$

If $q\mathbf{x}_1 = 0$,

$$\langle\sigma\mathbf{x}_1, \mathbf{x}_1\rangle = \langle\lambda^{-2}(2q-1)\mathbf{x}_1, \mathbf{x}_1\rangle = -\langle\lambda^{-2}\mathbf{x}_1, \mathbf{x}_1\rangle \leq 0.$$

On the other hand, $q\sigma = \sigma q^*$ if and only if $(2q-1)\sigma = \sigma(2q^*-1)$ and $\lambda^2 = (2q-1)\sigma$, so that $\lambda^2 = \sigma(2q^*-1)$. □

This result gives several characterizations of the set

$$\mathcal{Q}_\sigma = \{q \in \mathcal{Q} : \sigma(2q-1) \in G_2^+\}, \tag{11}$$

which plays a central role in this exposition. Notice also the immersion

$$\mathcal{Q}_\sigma \hookrightarrow G^+_{M_2}, \text{ defined by } q \mapsto (2q-1)\sigma \tag{12}$$

which will allow us to obtain in $\mathcal{Q}_\sigma$ several properties which hold in $G^+_{M_2}$.

Define

$$\pi : \mathcal{Q} \to \mathcal{P}\,, \quad \pi(q) = p,$$

if $q = \lambda_q p \lambda_q^{-1}$, where λ_q is the unique $\lambda \in G^+$ such that $2q - 1 = \lambda^2\sigma$. Clearly, π is a submersion and $\mathcal{Q}_\sigma = \pi^{-1}(p)$, where $2p - 1 = \sigma$. For later use, we recall that $\mathcal{Q}$ is a smooth submanifold of M_2 with tangent spaces given by

$$(T\mathcal{Q})_q = \{x \in M_2 : xq + qx = x\},$$

for $q \in \mathcal{Q}$. Also $\mathcal{P}$ is a smooth submanifold of $\mathcal{Q}$, and for $p \in \mathcal{P}$,

$$(T\mathcal{P})_p = \{x \in M_2 : xp + px = x \text{ and } x^* = x\}.$$

The fiber $\mathcal{Q}_\sigma$ is a smooth submanifold of $\mathcal{Q}$ (the map π is a submersion), and for $q \in \mathcal{Q}_\sigma$

$$(T\mathcal{Q}_\sigma)_q = \{x \in M_2 : xq + qx = x \text{ and } \sigma x^* = x\sigma\}.$$

For metrical properties of the smooth submersion $\pi : \mathcal{Q} \to \mathcal{P}$ and its tangent maps $(T\pi)_q$ the reader is referred to [12, 28].

2.4 The Unitary Group of a Symmetry

Every symmetry $\sigma \in M_2$ defines an $\mathcal{A}$-valued sesquilinear form θ_σ in $\mathcal{A}^2$:

$$\theta_\sigma : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}\,, \quad \theta_\sigma(\mathbf{x}, \mathbf{y}) = \langle \sigma\mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^*\sigma\mathbf{y}. \tag{13}$$

Every $a \in M_2$ admits a θ_σ-adjoint $a^{\sharp_\sigma} \in M_2$,

$$\theta_\sigma(a\mathbf{x}, \mathbf{y}) = \theta_\sigma(\mathbf{x}, a^{\sharp_\sigma}\mathbf{y}), \quad \text{i.e., } a^{\sharp_\sigma} = \sigma a^*\sigma. \tag{14}$$

The *unitary group* $\mathcal{U}_\sigma$ of θ_σ is

$$\begin{aligned}\mathcal{U}_\sigma &= \{a \in G_{M_2} : \theta_\sigma(a\mathbf{x}, a\mathbf{y}) = \theta_\sigma(\mathbf{x}, \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{A}^2\} \\ &= \{a \in G_{M_2} : \sigma a^* \sigma = a^{-1}\}. \end{aligned} \tag{15}$$

In this paper we shall study the group $\mathcal{U}_\sigma$ and some relevant homogeneous spaces, only for the special symmetries

$$\sigma = \rho := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma = \rho' := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the remaining of this section we focus in the case ρ. Explicitly,

$$\theta_\rho : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}\,, \theta_\rho(\mathbf{x}, \mathbf{y}) = x_1^* y_1 - x_2^* y_2 = \mathbf{x}^* \rho \mathbf{y}.$$

By (14), for $a \in M_2$ it holds that $a^{\sharp_\rho} = \rho a^* \rho = \begin{pmatrix} a_{11}^* & -a_{21}^* \\ -a_{12}^* & a_{22}^* \end{pmatrix}$. Therefore, $a \in \mathcal{U}_\rho$ if and only if $a \in G_{M_2}$ and satisfies $\rho a^* \rho = a^{-1}$; explicitly

$$\begin{cases} a_{11}^* a_{11} - a_{21}^* a_{21} = 1 \\ a_{11}^* a_{12} - a_{21}^* a_{22} = 0 \\ a_{12}^* a_{11} - a_{22}^* a_{21} = 0 \\ a_{22}^* a_{22} - a_{12}^* a_{12} = 1 \end{cases} \tag{16}$$

Remark 3 It can be shown that $\mathcal{U}_\rho$ is a Banach-Lie subgroup of G_{M_2} [31]. Its Banach-Lie algebra is given by

$$\mathfrak{U}_\rho = \{x \in M_2 : \rho x + x^* \rho = 0\} = \{\begin{pmatrix} b & x^* \\ x & c \end{pmatrix} : b, c \in \mathcal{A}_{ah}, x \in \mathcal{A}\},$$

where $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* + x = 0\}$.

Observe also that the Banach-Lie algebra of the subgroup $\mathcal{U}_\rho \cap \mathcal{U}_{M_2}$ is

$$(T(\mathcal{U}_\rho \cap \mathcal{U}_{M_2}))_1 = \{\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} : b, c \in \mathcal{A}_{ah}\}.$$

Remark 4 The subgroup $\mathcal{U}_\rho$ is closed for the adjoint operation. Indeed, $\rho a^* \rho a = 1$ implies that $1 = a(\rho a^* \rho a)a^{-1} = a\rho a^* \rho$. We get, thus, analogous identities as in (16). In particular, it holds that

$$a_{11} a_{11}^* - a_{12} a_{12}^* = 1,$$

and thus $a_{11}^* a_{11}, a_{11} a_{11}^* \geq 1$, a fortiori, $a_{11} \in G_{\mathcal{A}}$. Similarly, $a_{22} \in G_{\mathcal{A}}$.

It is not always easy to determine which are the elements of $\mathcal{U}_\rho$. However, some of its subgroups and subsets are more tractable. For instance, the subsets of unitary (resp. positive) elements of $\mathcal{U}_\rho$.

Proposition 3 *The subgroup of unitary elements of M_2 in $\mathcal{U}_\rho$, just considered, will be denoted by*

$$\mathbf{D}_\rho := \mathcal{U}_\rho \cap \mathcal{U}_{M_2} = \{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} : u_1, u_2 \in \mathcal{U}_{\mathcal{A}}\}. \tag{17}$$

Proof If $a \in \mathcal{U}_\rho \cap \mathcal{U}_{M_2}$, $\rho a^* \rho = a^{-1} = a^*$ and $\rho a \rho = a$, i.e., a and ρ commute. Thus a is a diagonal matrix. Being also a unitary matrix, its diagonal entries are unitary elements of $\mathcal{A}$. □

Proposition 4 *The set of positive elements in $\mathcal{U}_\rho$ is given by*

$$\mathcal{U}_\rho^+ := \mathcal{U}_\rho \cap G_2^+ = \{\begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix} : b \in \mathcal{A}\}. \tag{18}$$

Proof A simple computation shows that for any $b \in \mathcal{A}$ it holds that

$$\begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix}$$

belongs to $\mathcal{U}_\rho$. Conversely, if $a \in \mathcal{U}_\rho^+$, then $a_{11}, a_{22} \in G^+$ and $a_{12} = a_{21}^*$. Using the relations (16), we get

$$a_{11}^2 = 1 + a_{21}^* a_{21} \quad \text{and} \quad a_{22}^2 = 1 + a_{12}^* a_{12} = 1 + a_{21} a_{21}^*,$$

Therefore

$$a = \begin{pmatrix} (1+a_{21}^* a_{21})^{1/2} & a_{21}^* \\ a_{21} & (1+a_{21}a_{21}^*)^{1/2} \end{pmatrix}$$

as claimed. □

Therefore we get the following result. We omit the proof.

Proposition 5 *The map $\mathcal{A} \to \mathcal{U}_\rho^+$,*

$$b \mapsto \begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix}$$

is a diffeomorphism with inverse $\mathcal{U}_\rho^+ \to \mathcal{A}$, $\lambda \mapsto \lambda_{21}$.

We prove now that $\mathcal{U}_\rho^+$ consists of the positive factors of all reflections in $\mathcal{Q}_\rho$. Explicitly:

Proposition 6 *If* $\lambda_q = |2q-1|^{-1/2}$ *for* $q \in \mathcal{Q}$, *then*

$$\mathcal{U}_\rho^+ = \{\lambda_q : q \in \mathcal{Q}_\rho\} = \{((2q-1)\rho)^{1/2} : q \in \mathcal{Q}_\rho\}.$$

Proof Recall from Proposition 2, that $2q-1 = |2q-1|^{-1}\rho = \lambda_q^2\rho = \rho\lambda_q^{-2}$ and $q = \lambda_q p \lambda_q^{-1}$ (where $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1+\rho)$). Then

$$\rho\lambda_q^2\rho = \lambda_q^{-2}, \quad \rho\lambda_q\rho = \lambda_q^{-1} \text{ and } \lambda_q \in \mathcal{U}_\rho^+.$$

Conversely, we must prove that for any $\lambda \in \mathcal{U}_\rho^+$ there exists $q \in \mathcal{Q}_\rho$ such that $\lambda = \lambda_q$. By the last proposition

$$\lambda = \begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix}$$

and it is easy to see that

$$\lambda^{-1} = \begin{pmatrix} (1+b^*b)^{1/2} & -b^* \\ -b & (1+bb^*)^{1/2} \end{pmatrix}.$$

Then,

$$q = \lambda p \lambda^{-1} = \begin{pmatrix} 1+b^*b & -(1+b^*b)^{1/2}b^* \\ b(1+b^*b)^{1/2} & -bb^* \end{pmatrix}$$

and $2q-1 = \lambda(2p-1)\lambda^{-1} = \lambda\rho\lambda^{-1} = \lambda^2\rho$. Therefore, $q \in \mathcal{Q}_\rho$. By the uniqueness property of the polar decomposition, we obtain $\lambda = \lambda_q$. □

One has the following corollary of the proof:

Corollary 4 *If* $q \in \mathcal{Q}_\rho$, *then* $q_{11}, -q_{22} \in G^+$.

Proposition 7 *The map*

$$\mathcal{A} \to \mathcal{Q}_\rho, \ b \mapsto \begin{pmatrix} 1+b^*b & -(1+b^*b)^{1/2}b^* \\ b(1+b^*b)^{1/2} & -bb^* \end{pmatrix}$$

is a diffeomorphism whose inverse is

$$\mathcal{Q}_\rho \to \mathcal{A}, \ q \mapsto q_{21}q_{11}^{-1/2}.$$

By combining Propositions 5 and 7, we get:

Proposition 8 *The map*

$$\mathcal{Q}_\rho \to \mathcal{U}_\rho^+, \ q \mapsto ((2q-1)\rho)^{1/2}$$

is a diffeomorphism with inverse $\lambda \mapsto \lambda p \lambda^{-1}$.

Remark 5 Notice that the diffeomorphism above has the explicit matrix form

$$q \mapsto \begin{pmatrix} (1 + q_{11}^{-1/2} q_{21}^* q_{21} q_{11}^{-1/2})^{1/2} & q_{11}^{-1/2} q_{21}^* \\ q_{21} q_{11}^{-1/2} & (1 + q_{21} q_{11}^{-1} q_{21}^*)^{1/2} \end{pmatrix}. \tag{19}$$

The elements of $\mathcal{U}_\rho$ are known in different areas of mathematics as Möbius, linear fractional [32–34] or symplectic transformations [18, 26, 35].

2.5 $\mathcal{Q}_\rho$ as a Homogeneous Space of $\mathcal{U}_\rho$

In this section we prove that $\mathcal{Q}_\rho$ and $\mathcal{D}$ are diffeomorphic as reductive homogeneous spaces of $\mathcal{U}_\rho$, by means of an $\mathcal{U}_\rho$-equivariant diffeomorphism.

Recall that a smooth manifold M is a *homogeneous space* if it admits a transitive action of a Lie group G, such that M is diffeomorphic to the quotient G/K, where K is the Lie subgroup of all $g \in M$ such that $g \cdot m = m$, for m a fixed element of M (the *isotropy* group of m). If there exists a direct complement L of the Lie algebra of K in the Lie algebra of G, such that $gLg^{-1} \subset L$ for all $g \in K$, then M is called a *reductive homogeneous space* of G.

About these matters, we refer the reader to [17, 19–21, 24, 36–39].

We start by introducing a subset of $\mathcal{A}^2$ which plays a relevant role in what follows:

$$\mathcal{K}_\rho := \{\mathbf{x} \in \mathcal{A}^2 : \mathbf{x}^* \rho \mathbf{x} = 1,\ x_1 \in G_{\mathcal{A}}\}. \tag{20}$$

Notice that $\mathcal{K}_\rho$ consists of the elements of the unit sphere of θ_ρ whose first coordinate is invertible.

In order to identify the spaces $\mathcal{Q}_\rho$ and $\mathcal{D}$ we shall use projective methods. In fact, we present both spaces as subsets of the "projective line" associated with $\mathcal{A}^2$. For instance, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$ will serve to indicate the element $z \in \mathcal{D}$, $z = x_2 x_1^{-1}$ by means of its "homogeneous coordinates". Analogously, a point $q \in \mathcal{Q}_\rho$ is represented by $\mathbf{x} \in \mathcal{K}_\rho$ as the θ_ρ-orthogonal projection over the $\mathcal{A}$-submodule $[\mathbf{x}]$ spanned by $\mathbf{x}$.

We collect in the next proposition several useful facts which relate $\mathcal{K}_\rho$ with the group $\mathcal{U}_\rho$.

Proposition 9 1. *If* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$, *then* $\|x_2 x_1^{-1}\| < 1$.

2. *If* $a \in \mathcal{U}_\rho$ *and* $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *then* $a\mathbf{e}_1 \in \mathcal{K}_\rho$, *i.e. the first column of any* $a \in \mathcal{U}_\rho$ *belongs to* $\mathcal{K}_\rho$.
3. *If* $a \in \mathcal{U}_\rho$ *and* $\mathbf{x} \in \mathcal{K}_\rho$, *then* $a\mathbf{x} \in \mathcal{K}_\rho$.
4. *If* $\mathbf{x} \in \mathcal{K}_\rho$ *then there exists* $\mathbf{y} \in \mathcal{A}^2$ *such that* $\mathbf{x}^* \rho \mathbf{y} = 0$ *and* $\mathbf{y}^* \rho \mathbf{y} = -1$; *in particular* $a = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in \mathcal{U}_\rho$.

5. *The action of $\mathcal{U}_\rho$ on $\mathcal{K}_\rho$ by left multiplication (see 3. above) is transitive: $\mathcal{K}_\rho = \mathcal{U}_\rho \mathbf{x}$ for any $\mathbf{x} \in \mathcal{K}_\rho$.*

Proof 1. If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$ then $1 = x_1^* x_1 - x_2^* x_2$, and $x_1^* x_1 = 1 + x_2^* x_2$, and thus $x_2^* x_2 < x_1^* x_1$. Multiplying this inequality by $(x_1^*)^{-1}$ and x_1^{-1} (an operation which preserves strict inequalities because x_1 is invertible) we get

$$(x_1^*)^{-1} x_2^* x_2 x_1^{-1} = (x_2 x_1^{-1})^* (x_2 x_1^{-1}) < 1, \quad \text{i.e. } x_2 x_1^{-1} \in \mathcal{D}.$$

2. If $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{U}_\rho$, then by the identities (16) we get $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = a\mathbf{e}_1 \in \mathcal{K}_\rho$.
3. If $a \in \mathcal{U}_\rho$ and $\mathbf{x} \in \mathcal{K}_\rho$, then

$$(a\mathbf{x})^* \rho a \mathbf{x} = \theta_\rho(a\mathbf{x}, a\mathbf{x}) = \theta_\rho(\mathbf{x}, \mathbf{x}) = 1.$$

It remains to prove that $(a\mathbf{x})_1 \in G_{\mathcal{A}}$. Note that

$$a_{11} x_1 + a_{12} x_2 = a_{11}(1 + a_{11}^{-1} a_{12} x_2 x_1^{-1}) x_1.$$

Clearly $a_{11}, x_1 \in G_{\mathcal{A}}$, and $1 + a_{11}^{-1} a_{12} x_2 x_1^{-1}$ is also invertible because $\|a_{11}^{-1} a_{12}\| < 1$ by 2., and $\|x_2 x_1^{-1}\| < 1$ by 1.
4. If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$, then, for $\mathbf{y}' = \begin{pmatrix} (x_1^*)^{-1} x_2^* \\ 1 \end{pmatrix}$ it holds that

$$\theta_\rho(\mathbf{y}', \mathbf{y}') = (\mathbf{y}')^* \rho \mathbf{y}' = x_2 x_1^{-1} (x_1^*)^{-1} x_2^* - 1 < 0$$

by 1. Then, for $c = (1 - x_2 x_1^{-1} (x_1^*)^{-1} x_2^*)^{-1/2}$ and $\mathbf{y} = \mathbf{y}' c$ it holds $\theta_\rho(\mathbf{y}, \mathbf{y}) = -1$. A straightforward computation shows that $a = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in \mathcal{U}_\rho$. This also proves 5. □

One may rephrase these facts above to establish the following recipes:
Recipe 1: *How to construct (all) elements of $\mathcal{K}_\rho$.*
For $b \in \mathcal{A}$ it holds that $(1 + b^* b)^{1/2} \in G_{\mathcal{A}}$, so that $\mathbf{x} = \begin{pmatrix} (1 + b^* b)^{1/2} \\ b \end{pmatrix} \in \mathcal{K}_\rho$. Moreover, any other $\mathbf{x}' \in \mathcal{K}_\rho$ such that $x_2' = b$ has the form $\mathbf{x}' = \begin{pmatrix} u(1 + b^* b)^{1/2} \\ b \end{pmatrix}$ for some $u \in \mathcal{U}_{\mathcal{A}}$. In particular we have:

Proposition 10 *The map*

$$\Psi : \mathcal{A} \times \mathcal{U}_{\mathcal{A}} \to \mathcal{K}_\rho \,, \quad \Psi(b, u) = \begin{pmatrix} u(1 + b^* b)^{1/2} \\ b \end{pmatrix}$$

is a diffeomorphism with inverse

$$\Psi^{-1} : \mathcal{K}_\rho \to \mathcal{A} \times \mathcal{U}_\mathcal{A}\,, \quad \Psi^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1(1 + x_2^* x_2)^{-1/2} \end{pmatrix}.$$

Recipe 2: *How to construct $a \in \mathcal{U}_\rho$ whose first column is a fixed $\mathbf{x} \in \mathcal{K}_\rho$.*

If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$, then for $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (x_1^*)^{-1} x_2^* \\ 1 \end{pmatrix} c$, $c = (1 - x_2 x_1^{-1}$ $(x_1^*)^{-1} x_2^*)^{-1/2}$, it holds that $a = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in \mathcal{U}_\rho$.

The next theorem is a useful characterization of $\mathcal{Q}_\rho$ by means of elements of $\mathcal{K}_\rho$; it will be frequently used in the sequel. First we need to establish a few facts.

Definition 2 If $\mathbf{x} \in \mathcal{K}_\rho$, denote

$$p_\mathbf{x} = \mathbf{x}\mathbf{x}^* \rho \in M_2. \tag{21}$$

Lemma 1 *For $\mathbf{x} \in \mathcal{K}_\rho$ it holds that $p_\mathbf{x} \in \mathcal{Q}$ and $R(p_\mathbf{x}) = [\mathbf{x}] := \{\mathbf{x}a : a \in \mathcal{A}\}$. In particular, this latter space is a closed and complemented submodule of $\mathcal{A}^2$.*

Proof First, note that for $\mathbf{x} \in \mathcal{K}_\rho$ it holds that

$$p_\mathbf{x}^2 = \mathbf{x}\mathbf{x}^* \rho \mathbf{x}\mathbf{x}^* \rho = p_\mathbf{x},$$

because $\mathbf{x}^* \rho \mathbf{x} = 1$. By the same argument, $p_\mathbf{x}\mathbf{x} = \mathbf{x}$. Since $p_\mathbf{x} \in \mathcal{Q}$, its range $R(p_\mathbf{x})$ is a closed and complemented submodule of $\mathcal{A}^2$, which contains $\mathbf{x}$, thus $[\mathbf{x}] \subset R(p_\mathbf{x})$. Conversely, if $\mathbf{z} \in R(p_\mathbf{x})$, then

$$\mathbf{z} = p_\mathbf{x}\mathbf{z} = \mathbf{x}\mathbf{x}^* \rho = \mathbf{x}(\mathbf{x}^* \rho \mathbf{z}) \in [\mathbf{x}].$$

□

Lemma 2 *For $\mathbf{x}, \mathbf{x}' \in \mathcal{K}_\rho$, the following conditions are equivalent:*

1. *$p_\mathbf{x} = p_{\mathbf{x}'}$;*
2. *$[\mathbf{x}] = [\mathbf{x}']$;*
3. *there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}' = \mathbf{x}u$*

Proof If $p_\mathbf{x} = p_{\mathbf{x}'}$, then $[\mathbf{x}] = R(p_\mathbf{x}) = R(p_{\mathbf{x}'}) = [\mathbf{x}']$. Then $\mathbf{x} = p_{\mathbf{x}'}\mathbf{x}$ and $\mathbf{x}' = p_\mathbf{x}\mathbf{x}'$. We claim that $u = \mathbf{x}^* \rho \mathbf{x}' \in \mathcal{U}_\mathcal{A}$. Indeed

$$uu^* = \mathbf{x}^* \rho \mathbf{x}'(\mathbf{x}')^* \rho \mathbf{x} = \mathbf{x}^* \rho p_{\mathbf{x}'} \mathbf{x} = \mathbf{x}^* \rho \mathbf{x} = 1.$$

Similarly, $u^* u = 1$. Note that $\mathbf{x}u = \mathbf{x}\mathbf{x}^* \rho \mathbf{x}' = p_\mathbf{x}\mathbf{x}' = \mathbf{x}'$.

If $\mathbf{x}' = \mathbf{x}u$ for some $u \in \mathcal{U}_\mathcal{A}$, then $p_{\mathbf{x}'} = \mathbf{x}u(\mathbf{x}u)^* \rho = \mathbf{x}uu^* \mathbf{x}^* \rho = p_\mathbf{x}$. □

Theorem 3 *$\mathcal{Q}_\rho$ coincides with the set $\{p_\mathbf{x} : \mathbf{x} \in \mathcal{K}_\rho\}$.*

Proof First we show that $p_{\mathbf{x}} \in \mathcal{Q}_\rho$ for any $\mathbf{x} \in \mathcal{K}_\rho$. We need to prove that $p_{\mathbf{x}}^\sharp = p_{\mathbf{x}}$, and θ_ρ is positive in $R(p_{\mathbf{x}})$ and negative in $N(p_{\mathbf{x}})$. We get

$$p_{\mathbf{x}}^\sharp = \rho p_{\mathbf{x}}^* \rho = \rho(\mathbf{x}\mathbf{x}^*\rho)^*\rho = \rho\rho\mathbf{x}\mathbf{x}^*\rho = \mathbf{x}\mathbf{x}^*\rho = p_{\mathbf{x}},$$

since $\rho^2 = 1$. Next, note that

$$\theta_\rho(p_{\mathbf{x}}\mathbf{y}, p_{\mathbf{x}}\mathbf{y}) = (p_{\mathbf{x}}\mathbf{y})^*\rho(p_{\mathbf{x}}\mathbf{y}) = \mathbf{y}^*\rho\mathbf{x}\mathbf{x}^*\rho\mathbf{x}\mathbf{x}^*\rho\mathbf{y} = \mathbf{y}^*\rho p_{\mathbf{x}}\mathbf{y} = \mathbf{y}^*\rho p_{\mathbf{x}}^2\mathbf{y} = \mathbf{y}^*\rho\mathbf{x}\mathbf{x}^*\rho\mathbf{y} \geq 0.$$

If $\mathbf{z} \in N(p_{\mathbf{x}})$ then $\mathbf{x}\mathbf{x}^*\rho\mathbf{z} = 0$ and, since $x_1 \in G_{\mathcal{A}}$ and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathbf{x}^*\rho\mathbf{z}$$

it must be $\mathbf{x}^*\rho\mathbf{z} = 0$, i.e., $x_1^* z_1 = x_2^* z_2$, or $z_1 = (x_1^*)^{-1}x_2^* z_2$. Then,

$$\mathbf{z}^*\rho\mathbf{z} = z_2^* x_2 x_1^{-1}(x_1^*)^{-1}x_2^* z_2 - z_2^* z_2 = z_2^*(aa^* - 1)z_2,$$

where $a = x_2 x_1^{-1}$. Since, by Proposition 9, part 1, $\|a\| < 1$, it follows that θ_ρ is negative in $N(p_{\mathbf{x}})$, and $p_{\mathbf{x}} \in \mathcal{Q}_\rho$ as claimed. For the converse inclusion, recall from Proposition 7 that every $q \in \mathcal{Q}_\rho$ has the form

$$q = \begin{pmatrix} 1 + b^*b & -(1+b^*b)^{1/2}b^* \\ b(1+b^*b)^{1/2} & -bb^* \end{pmatrix}$$

for some $b \in \mathcal{A}$. But $\mathbf{x} = \begin{pmatrix} (1+b^*b)^{1/2}b^* \\ b \end{pmatrix} \in \mathcal{K}_\rho$, and $p_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*\rho = q$, which finishes the proof. □

Corollary 5 *If $q = p_{\mathbf{x}}$ for some $\mathbf{x} \in \mathcal{K}_\rho$, the formula in Remark 5 is much simpler:*

$$p_{\mathbf{x}} \mapsto \begin{pmatrix} (x_1x_1^*)^{1/2} & (x_1x_1^*)^{-1/2}x_1x_2^* \\ x_2x_1^*(x_1x_1^*)^{-1/2} & (1+x_2x_2^*)^{1/2} \end{pmatrix}.$$

Theorem 4 *1. The map*

$$\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho \quad \varphi_\rho(\mathbf{x}) = p_{\mathbf{x}}$$

is a smooth principal fibration, with group $\mathcal{U}_{\mathcal{A}}$ and with a global cross section

$$\sigma_\rho : \mathcal{Q}_\rho \to \mathcal{K}_\rho\,, \quad \sigma_\rho(q) = \lambda_q \mathbf{e}_1 = \begin{pmatrix} q_{11}^{1/2} \\ q_{21}q_{11}^{-1/2} \end{pmatrix}.$$

2. The map

$$\Phi : \mathcal{K}_\rho \to \mathcal{Q}_\rho \times \mathcal{U}_{\mathcal{A}}\,, \quad \Phi(\mathbf{x}) = (p_{\mathbf{x}}, \mathbf{x}^*\rho\sigma_\rho(p_{\mathbf{x}}))$$

is a diffeomorphism with inverse

$$\Phi^{-1}(q,u) = \sigma_\rho(q)u^*.$$

Proof Theorem 3 says that φ_ρ is onto. It is clearly smooth. In order to verify that $\lambda_q \mathbf{e}_1 \in \mathcal{K}_\rho$, note that

$$(\lambda_q \mathbf{e}_1)^* \rho \lambda_q \mathbf{e}_1 = \mathbf{e}_1^* \lambda_q \rho \lambda_q \mathbf{e}_1 = \mathbf{e}_1^* \rho \mathbf{e}_1 = 1$$

because $\lambda_q \rho \lambda = \rho$, by Proposition 2. The fact that $(\lambda_q \mathbf{e}_1)_1$ is invertible follows from the invertibility of q_{11} (Corollary 4). The proof that σ_ρ is a cross section for φ_ρ follows from the identities

$$\varphi_\rho(\sigma_\rho(q)) = \varphi_\rho(\lambda_q \mathbf{e}_1) = \lambda_q \mathbf{e}_1 \mathbf{e}_1^* \lambda_q \rho = \lambda_q p \rho \lambda_q^{-1} = \lambda_q p \lambda_q^{-1} = q,$$

because $\mathbf{e}_1 \mathbf{e}_1^* = p$ and $p\rho = p$.

For the second part we must show that $\mathbf{x}^* \rho \lambda_{p_\mathbf{x}} \mathbf{e}_1 \in \mathcal{U}_\mathcal{A}$. But

$$\mathbf{x}^* \rho \lambda_{p_\mathbf{x}} \mathbf{e}_1 \mathbf{e}_1^* \lambda_{p_\mathbf{x}} \rho \mathbf{x} = \mathbf{x}^* \rho \lambda_{p_\mathbf{x}} p \rho \lambda_{p_\mathbf{x}}^{-1} \mathbf{x} = \mathbf{x}^* \rho \lambda_{p_\mathbf{x}} p \lambda_{p_\mathbf{x}}^{-1} \mathbf{x} = \mathbf{x}^* \rho p_\mathbf{x} \mathbf{x} = \mathbf{x}^* \rho \mathbf{x} = 1.$$

The equality $(\mathbf{e}_1^* \lambda_{p_\mathbf{x}} \rho \mathbf{x})(\mathbf{x}^* \rho \lambda_{p_\mathbf{x}} \mathbf{e}_1) = 1$ follows similarly. Finally, a straightforward computation shows that these maps are each other inverses. □

Remark 6 One should notice that, with the current assumptions and notations, $\mathbf{x}^* \rho \sigma_\rho(p_\mathbf{x}) = x_1^* (x_1 x_1^*)^{-1/2}$

The *tautological bundle* of $\mathcal{Q}_\rho$ is defined by

$$\{(q, \mathbf{x}) \in \mathcal{Q}_\rho \times \mathcal{A}^2 : \mathbf{x} \in R(q)\}$$

with the projection $(q, \mathbf{x}) \mapsto q$. We get the diagram

$$\begin{array}{ccc} \mathcal{K}_\rho & \hookrightarrow & \{(a, \mathbf{x}) \in \mathcal{Q}_\rho \times \mathcal{A}^2 : \mathbf{x} \in R(q)\} \\ & \searrow & \downarrow \\ & & \mathcal{Q}_\rho \end{array} \tag{22}$$

where the horizontal arrow is given by $\mathbf{x} \mapsto (p_\mathbf{x}, \mathbf{x})$ $(\mathbf{x} \in \mathcal{K}_\rho)$. A corollary of Theorem 3 is that $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$ is the principal bundle with group $\mathcal{U}_\mathcal{A}$ associated to the tautological bundle. We shall pursue this matter later (Sect. 1.1 in Chap. 4).

Proposition 11 *1. $\mathcal{U}_\rho$ acts on $\mathcal{Q}_\rho$ by similarity:*

$$\mathcal{U}_\rho \times \mathcal{Q}_\rho \to \mathcal{Q}_\rho, \ (a, q) \mapsto aqa^{-1}.$$

2. The action is transitive.

3. For every $q_0 \in \mathcal{Q}_\rho$, the map

$$\pi_{q_0} : \mathcal{U}_\rho \to \mathcal{Q}_\rho,\ \pi_{q_0}(a) = aq_0a^{-1}$$

is smooth and onto. Moreover, π_{q_0} admits a global cross section.

Proof A simple computation shows that $aqa^{-1} \in \mathcal{Q}$ for every $q \in \mathcal{Q}_\rho$; we claim that $aqa^{-1} \in \mathcal{Q}_\rho$, i.e., $(2aqa^{-1})\rho \in G^+_{M_2}$. Indeed, by (14), it holds $a^{-1}\rho = \rho a^*$ and, then,

$$(2aqa^{-1} - 1)\rho = a(2q - 1)a^{-1}\rho = a(2q - 1)\rho a^*\rho\rho = a(2q - 1)\rho a^* \geq 0.$$

For the transitivity it suffices to show that for every $q \in \mathcal{Q}_\rho$ there exists $a \in \mathcal{U}_\rho$ such that $apa^{-1} = q$, where $p = \frac{1}{2}(1 + \rho) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We proved in Proposition 2 that $q = \lambda_q p \lambda_q^{-1}$, where $\lambda_q \in \mathcal{U}_\rho \cap G^+_{M_2}$. The surjectivity follows from the existence of a global cross section, which we prove next. For π_p it has already been proved (the cross section is given by $q \mapsto \lambda_q$). One can obtain a cross section for any other π_{q_0} by translating this one with the group action. □

Corollary 6 *For $q \in \mathcal{Q}_\rho$, the isotropy group H_q of q,*

$$H_q = \{a \in \mathcal{U}_\rho : aqa^{-1} = q\}$$

is a closed Lie-Banach subgroup of $\mathcal{U}_\rho$ and the bijection $\mathcal{U}_\rho/H_q \leftrightarrow \mathcal{Q}_\rho$ is a diffeomorphism. In other terms, $\mathcal{Q}_\rho$ is a homogeneous space of $\mathcal{U}_\rho$.

The proof is straightforward. The cross section of the above proposition induces the inverse diffeomorphism.

Recall that a homogeneous space $M = G/K$ is reductive if the Lie algebra of G decomposes as the Lie algebra of K plus a supplement L with the property that $hLh^{-1} \subset L$ for every $h \in K$.

We note now that $\mathcal{Q}_\rho$ is a reductive homogeneous space of $\mathcal{U}_\rho$. We proved before (Remark 3) that the Lie algebra of $\mathcal{U}_\rho$ is

$$\mathfrak{u}_\rho = \{x \in M_2 : \rho x^* \rho = -x\} = \{\begin{pmatrix} b & c^* \\ c & d \end{pmatrix} : b, c, d \in \mathcal{A},\ b + b^* = 0,\ d + d^* = 0\}.$$

The isotropy group of p in $\mathcal{U}_\rho$ is

$$\mathcal{I}_p = \{a \in \mathcal{U}_\rho : ap = pa\} = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \in \mathcal{U}_\mathcal{A}\}.$$

Then its Lie algebra is

$$\mathfrak{i}_p = \{\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} : b + b^* = d + d^* = 0\},$$

and a natural supplement for $\mathfrak{i}_p$ in $\mathfrak{u}_\rho$ is

$$\mathbf{H}_p = \{\begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} : c \in \mathcal{A}\}. \tag{23}$$

It is fairly obvious that $a\mathbf{H}_p a^{-1} \subset \mathbf{H}_p$ for every $a \in \mathcal{I}_p$.

It follows then that the unique geodesic δ in $\mathcal{Q}_\rho$ with $\delta(0) = p$ and $\dot{\delta}(0) = X \in \mathbf{H}_p$ is

$$\delta(t) = e^{tX} \cdot p = e^{\frac{t}{2}X} p\, e^{-\frac{t}{2}X}. \tag{24}$$

Recall from (12) the immersion

$$\Phi : \mathcal{Q}_\rho \to G^+_{M_2}\,, \quad q :\mapsto ((2q-1)\rho)^{1/2};$$

we have seen before that $im\ \Phi = \mathcal{U}^+_\rho$.

We shall prove later that this immersion preserves the relevant geometrical structures. In particular, it translates geodesics of $\mathcal{Q}_\rho$ into geodesics of $G^+_{M_2}$. Conversely, every geodesic of $G^+_{M_2}$ contained in $\mathcal{U}^+_\rho$ is the image of a geodesic in $\mathcal{Q}_\rho$. Using this translation we are able to compute in an explicit way all geodesics of $\mathcal{Q}_\rho$.

If $q_0, q_1 \in \mathcal{Q}_\rho$, then

$$\mu_0 = ((2q_0 - 1)\rho)^{1/2}\,, \quad \mu_1 = ((2q_1 - 1)\rho)^{1/2} \tag{25}$$

belong to $G^+_{M_2}$ and, therefore, they are joined by a unique geodesic μ in $G^+_{M_2}$. Explicitly, the curve

$$\mu : [0,1] \to G^+_{M_2}\,, \quad \mu(t) = \mu_0^{1/2} \left(\mu_0^{-1/2} \mu_1 \mu_0^{-1/2}\right)^t \mu_0^{1/2}$$

is the unique geodesic in $G^+_{M_2}$ such that $\mu(0) = \mu_0$ and $\mu(1) = \mu_1$ (recall (3) in Sect. 2.1). By definition, $\mu_0, \mu_1 \in im\ \Phi = \mathcal{U}^+_\rho$; it is an easy exercise to show that $\mu(t) \in \mathcal{U}^+_\rho$ for all $t \in [0,1]$ (for all $t \in \mathbb{R}$, in fact). From the polar decompositions

$$(2q_0 - 1) = (2q_0 - 1)\rho\rho = \mu_0^2\rho,\ (2q_1 - 1) = (2q_1 - 1)\rho\rho = \mu_1^2\rho$$

and the fact (Proposition 2) that $q_0 = \mu_0 p \mu_0^{-1}$ and $q_1 = \mu_1 p \mu_1^{-1}$ we see that

$$q(t) = \mu(t) p \mu^{-1}(t) \tag{26}$$

is the geodesic in $\mathcal{Q}_\rho$ joining q_0 and q_1.

Remark 7 It is apparent that $\mathcal{U}^+_\rho$ is also a reductive homogeneous space of $\mathcal{U}_\rho$. We do not pursue its study here, since its geometry is that of $\mathcal{Q}_\rho$.

2.6 $\mathcal{D}$ as a Homogeneous Space of $\mathcal{U}_\rho$

Our next step is to show that the Poincaré disk

$$\mathcal{D} = \{z \in \mathcal{A} : z^*z < 1\} = \{z \in \mathcal{A} : \|z\| < 1\}$$

is a reductive homogeneous space of $\mathcal{U}_\rho$, equivariantly diffeomorphic to $\mathcal{Q}_\rho$.

Recall from Proposition 9 that $x_2x_1^{-1} \in \mathcal{D}$ for every $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$.

Proposition 12 *The map*

$$\varphi_D : \mathcal{K}_\rho \to \mathcal{D}\,, \quad \varphi_D(\mathbf{x}) = x_2x_1^{-1}$$

is a smooth submersion with a smooth global cross section.

Proof If $z \in \mathcal{D}$,

$$\theta_\rho(\begin{pmatrix} 1 \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ z \end{pmatrix}) = \begin{pmatrix} 1 & z^* \end{pmatrix} \rho \begin{pmatrix} 1 \\ z \end{pmatrix} = 1 - z^*z \in G^+.$$

Then $\mathbf{x}_z := \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^*z)^{-1/2} \in \mathcal{K}_\rho$, and $\varphi_D(\mathbf{x}_z) = z$. Thus $\sigma_D(z) = \mathbf{x}_z$ defines a smooth global cross section for φ_D. □

Notice that $\sigma_D(\varphi_D(\mathbf{x})) = \mathbf{x}_{x_2x_1^{-1}} = \begin{pmatrix} 1 \\ x_2x_1^{-1} \end{pmatrix} (x_1x_1^*)^{1/2}$.

We have the following proposition, with straightforward proof:

Proposition 13 *If $\mathbf{x} \in \mathcal{K}_\rho$ then $\mathbf{x}^*\rho\sigma_D(x_2x_1^{-1}) \in \mathcal{U}_\mathcal{A}$, and the map*

$$\Phi_D : \mathcal{K}_\rho \to \mathcal{D} \times \mathcal{U}_\mathcal{A},\ \Phi_D(\mathbf{x}) = (x_2x_1^{-1}, \mathbf{x}^*\rho\sigma_D(x_2x_1^{-1}))$$

is a diffeomorphism with inverse

$$\Phi_D^{-1}(z, u) = \sigma_D(z)u^* = \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^*z)^{-1/2}u^*.$$

Definition 3 Define the action of $\mathcal{U}_\rho$ on $\mathcal{D}$ as follows: for $a \in \mathcal{U}_\rho$ and $z \in \mathcal{D}$, put

$$a \cdot z := \varphi_D(a\sigma_D(z)) = \varphi_D(a \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^*z)^{-1/2}) = (a_{21} + a_{22}z)(a_{11} + a_{12}z)^{-1}.$$

Proposition 14 *The action of $\mathcal{U}_\rho$ on $\mathcal{D}$ is transitive.*

Proof Note first that $a_{11} + a_{12}z$ is invertible because $a \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^*z)^{-1/2} \in \mathcal{K}_\rho$, and therefore, its first coordinate $(a_{11} + a_{12}z)(1 - z^*z)^{-1/2}$ is invertible. The fact

that $(ba)\cdot z = b\cdot(a\cdot z)$ follows easily from straightforward computations. So, we have defined a left action of $\mathcal{U}_\rho$ on $\mathcal{D}$. Let us show that it is transitive. For this, it suffices to prove that for any $z\in\mathcal{D}$, there exists $a\in\mathcal{U}_\rho$ such that $a\cdot 0 = z$. With a similar argument as in Proposition 9, part 4, it easy to show that

$$g_z = \begin{pmatrix} (1-z^*z)^{-1/2} & (1-z^*z)^{-1/2}z^* \\ z(1-z^*z)^{-1/2} & (1-zz^*)^{-1/2} \end{pmatrix} \in \mathcal{U}_\rho \tag{27}$$

and $g_z\cdot 0 = z$. Thus, the action is transitive. □

For each $z_0\in\mathcal{D}$, consider

$$\pi_{z_0} : \mathcal{U}_\rho \to \mathcal{D}\,,\quad \pi_{z_0}(a) = a\cdot z_0.$$

Theorem 5 *$\mathcal{D}$ is a reductive homogeneous space of $\mathcal{U}_\rho$.*

Proof Consider $z_0 = 0$. The isotropy subgroup $\mathcal{I}_0 = \{a\in\mathcal{U}_\rho : a\cdot 0 = 0\}$ is

$$\mathcal{I}_0 = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v\in\mathcal{U}_\mathcal{A}\}.$$

Indeed, $a\cdot 0 = a_{21}a_{11}^{-1}$, so $a\cdot 0 = 0$ which means that $a_{21} = 0$. From the definition of $\mathcal{U}_\rho$ (see the second equation in (16)) it readily follows that $a_{12} = 0$, and then $a_{11}, a_{22}\in\mathcal{U}_\mathcal{A}$. Then, as in the case of $\mathcal{Q}_\rho$, the Lie algebra of $\mathcal{I}_0$ is

$$\mathfrak{i}_0 = \{\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} : b^* = -b,\ d^* = -d\}$$

and a supplement for $\mathfrak{i}_0$ in $\mathfrak{u}_\rho$ is

$$\mathbf{H}_0 = \{\begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} : c\in\mathcal{A}\}$$

which is a reductive complement. □

As in the case of $\mathcal{Q}_\rho$, the geodesics of the affine connection of $\mathcal{D}$ starting at 0 are, exactly, those of the form $\delta(t) = e^{tX}\cdot 0$, with $X\in\mathbf{H}_0$; the geodesics starting at $z\in\mathcal{D}$ are $g_z\cdot(\delta(t))$.

The next result identifies $\mathcal{Q}_\rho$ and $\mathcal{D}$ as reductive homogeneous spaces of $\mathcal{U}_\rho$.

Theorem 6 *There exists a diffeomorphism $\alpha_D : \mathcal{Q}_\rho\to\mathcal{D}$ which makes the following diagram commutative*

$$\begin{array}{ccc} & \mathcal{K}_\rho & \\ {\scriptstyle\varphi_\rho}\swarrow & & \searrow{\scriptstyle\varphi_D} \\ \mathcal{Q}_\rho & \xrightarrow{\quad\alpha_D\quad} & \mathcal{D}, \end{array} \tag{28}$$

All maps in the diagram are $\mathcal{U}_\rho$-equivariant, i.e., for every $\mathbf{x} \in \mathcal{K}_\rho$, $a \in \mathcal{U}_\rho$ and $q \in \mathcal{Q}_\rho$ it holds that

$$\varphi_\rho(a\mathbf{x}) = ap_{\mathbf{x}}a^{-1}\,,\quad \varphi_D(a\mathbf{x}) = a \cdot (x_2x_1^{-1}) \text{ and } \alpha_D(ap_{\mathbf{x}}a^{-1}) = p_{a\mathbf{x}}.$$

Proof Notice that if $\mathbf{x}, \mathbf{y} \in \mathcal{K}_\rho$ and $p_{\mathbf{x}} = p_{\mathbf{y}}$, then $x_2x_1^{-1} = y_2y_1^{-1}$. Indeed, if $p_{\mathbf{x}} = p_{\mathbf{y}}$ then there exists $u \in \mathcal{U}_{\mathcal{A}}$ such that $\mathbf{y} = \mathbf{x}u$, and then $y_2y_1^{-1} = x_2u(x_1u)^{-1} = x_2x_1^{-1}$. Thus, we can define

$$\alpha_D : \mathcal{Q}_\rho \to \mathcal{D}\,,\quad \alpha_D(p_{\mathbf{x}}) = x_2x_1^{-1}. \tag{29}$$

This proves the commutativity. Note that $\alpha_D(q) = q_{21}q_{11}^{-1}$ is an equivalent definition for α_D. It is also a routine exercise to check the identities

$$\alpha_D \circ (\varphi_D \circ \sigma_D) = I_D \text{ and } (\varphi_D \circ \sigma_D) \circ \alpha_D = I_{\mathcal{Q}_\rho}.$$

They show that $\alpha_D^{-1} = \varphi_D \circ \sigma_D : z \mapsto p_{\mathbf{x}_z}$. It remains to prove that φ_ρ, φ_D and α_D are $\mathcal{U}_\rho$-equivariant.

If $\mathbf{x} \in \mathcal{K}_\rho$ and $a \in \mathcal{U}_\rho$, then

- $\varphi_\rho(a\mathbf{x}) = p_{a\mathbf{x}} = a\mathbf{x}\mathbf{x}^*a^*\rho = a\mathbf{x}\mathbf{x}^*(\rho a^{-1}\rho)\rho = a\mathbf{x}\mathbf{x}^*\rho a^{-1} = ap_{\mathbf{x}}a^{-1}$.
- $\varphi_D(a\mathbf{x}) = (a\mathbf{x})_2(a\mathbf{x})_1^{-1} = (a_{21}x_1 + a_{22}x_2)(a_{11}x_1 + a_{12}x_2)^{-1}$ $= (a_{21} + a_{22}x_2x_1^{-1})x_1x_1^{-1}(a_{11} + a_{12}x_2x_1^{-1})^{-1} = a \cdot x_2x_1^{-1}$.
- $\alpha_D(ap_{\mathbf{x}}a^{-1}) = \alpha_D(p_{a\mathbf{x}}) = (a\mathbf{x})_2(a\mathbf{x})_1^{-1} = a \cdot (x_2x_1^{-1}) = a \cdot \alpha_D(p_{\mathbf{x}})$.

□

2.7 Geodesics of $\mathcal{D}$

We continue here the study of the geodesics of $\mathcal{D}$.

It is easy to see that for any $z \in \mathcal{D}$, the unique geodesic δ of $\mathcal{D}$ such that $\delta(0) = 0$ and $\delta(1) = z$ is given by

$$\delta(t) = e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \cdot 0\,,\quad \text{where } \alpha = z\sum_{k=0}^{\infty} \frac{1}{2k+1}(z^*z)^k. \tag{30}$$

Note here that $\sum_{k=0}^{\infty} \frac{t^{2k+1}}{2k+1}$ is the Taylor series of the function $f(t) = \frac{1}{2}\log(\frac{1+t}{1-t})$, in the interval $(-1, 1)$. We shall compute below the explicit form of δ in $\mathcal{D}$, joining 0 and $z \in \mathcal{D}$ at time $t = 1$:

Lemma 3 *Given $z \in \mathcal{D}$, the unique geodesic δ of $\mathcal{D}$ with $\delta(0) = 0$ and $\delta(1) = z$ is given by*

$$\delta(t) = \omega \tanh(t|\alpha|), \tag{31}$$

where α is given in (30) above, and ω is the partial isometry in the polar decomposition of α: $\alpha = \omega|\alpha|$, performed in $\mathcal{A}^{**}$.

Proof Straightforward computations show that the even and odd powers of $\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$ are, respectively,

$$\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}^{2k} = \begin{pmatrix} (\alpha^*\alpha)^k & 0 \\ 0 & (\alpha\alpha^*)^k \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}^{2k+1} = \begin{pmatrix} 0 & \alpha^*(\alpha\alpha^*)^k \\ \alpha(\alpha^*\alpha)^k & 0 \end{pmatrix}.$$

We are interested in the first column of $e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}}$, which is

$$\begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}(\alpha^*\alpha)^k \\ \alpha \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(\alpha^*\alpha)^k \end{pmatrix} = \begin{pmatrix} \cosh(t|\alpha|) \\ \omega \sinh(t|\alpha|) \end{pmatrix}.$$

Then,

$$\delta(t) = \omega \sinh(t|\alpha|)(\cosh(t|\alpha|))^{-1} = \omega \tanh(t|\alpha|).$$

□

Notice that $\omega \in \mathcal{A}^{**}$ need not belong to $\mathcal{A}$. However, $\delta(t) \in \mathcal{A}$ for all t. Concerning partial isometries in C*-algebras, see [40].

Let us relate the polar decompositions of z and α.

Proposition 15 *If $z \in \mathcal{D}$ and α as in (30), then*

$$\alpha = \frac{1}{2}\omega(\log(1+|z|) - \log(1-|z|)), \quad \textit{and} \quad z = \omega|z|,$$

i.e., the partial isometry $\omega \in \mathcal{A}^{**}$ is the same for α and z.

Proof First, note that

$$|\alpha|^2 = \alpha^*\alpha = \sum_{k=0}^{\infty} \frac{1}{2k+1}(z^*z)^k z^*z \sum_{k=0}^{\infty} \frac{1}{2k+1}(z^*z)^k = \left(\sum_{k=0}^{\infty} \frac{1}{2k+1}|z|^{2k+1}\right)^2,$$

i.e., $|\alpha| = \frac{1}{2}(\log(1+|z|) - \log(1-|z|))$. Next, consider $z = v|z|$ the polar decomposition of z. Note that, since $\alpha = z\sum_{k=0}^{\infty} \frac{1}{2k+1}|z|^{2k}$, we have that

$$\alpha = v|z| \sum_{k=0}^{\infty} \frac{1}{2k+1}|z|^{2k} = v|\alpha|.$$

Thus, in order to prove our claim, it suffices to show that both partial isometries v, ω have the same initial and final spaces (the result follows, then, by the uniqueness

property of the polar decomposition). The partial isometry v maps $N(z)^\perp = N(|z|)^\perp$ onto $\overline{R(z)}$, whereas ω maps $N(\alpha)^\perp$ onto $\overline{R(\alpha)}$. If $z\xi = 0$, then

$$\alpha\xi = z\sum_{k=0}^{\infty}\frac{1}{2k+1}(z^*z)^k\xi = \sum_{k=0}^{\infty}\frac{1}{2k+1}(zz^*)^k z\xi = 0,$$

i.e., $N(z) \subset N(\alpha)$. Conversely, in the last expression of α, $\alpha = \sum_{k=0}^{\infty}\frac{1}{2k+1}|z^*|^{2k}z$; observe that

$$\sum_{k=0}^{\infty}\frac{1}{2k+1}|z^*|^{2k} = g(|z^*|),$$

where $g(t) = \frac{1}{2t}(\log(1+t) - \log(1-t))$ (which can be extended continuously as $g(0) = 1$), is defined in $\sigma(|z^*|) \subset [0,1)$, and is nonvanishing there. Therefore, $g(|z^*|)$ is invertible. Thus, $\alpha\xi = 0$ implies $z\xi = 0$. Again, using the function g, we get $\alpha = zg(|z|)$, and thus $R(z) = R(\alpha)$. □

Corollary 7 *The exponential map* $\mathbf{Exp}_0$ *of* $\mathcal{D}$ *at* 0*, and its inverse* $\mathbf{Log}_0$ *can be written explicitly as follows: if* $z = \omega|z| \in \mathcal{D}$

$$\mathbf{Log}_0 : \mathcal{D} \to (T\mathcal{D})_0\ ,\ \mathbf{Log}_0(z) = \frac{1}{2}\omega\log\left((1+|z|)(1-|z|)^{-1}\right).$$

If $\alpha = \omega|\alpha| \in \mathcal{A} \simeq (T\mathcal{D})_0$*, then*

$$\mathbf{Exp}_0 : (T\mathcal{D})_0 \to \mathcal{D}\ ,\ \mathbf{Exp}_0(\alpha) = \omega\tanh(|\alpha|).$$

In particular, if $\alpha = \mathbf{Log}_0(z)$ *(or, equivalently,* $z = \mathbf{Exp}_0(\alpha)$*), then* z *and* α *have the same partial isometry in the polar decomposition. Also,*

$$|\mathbf{Exp}_0(\alpha)| = \tanh(|\alpha|)\ \ \textit{and}\ \ |\mathbf{Log}_0(z)| = \frac{1}{2}\log\left((1+|z|)(1-|z|)^{-1}\right). \tag{32}$$

2.8 $\mathcal{H}$ as a Reductive Homogeneous Space

Consider now the symmetry

$$\rho' := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{33}$$

the $\mathcal{A}$-valued form $\theta_{\rho'}$

$$\theta_{\rho'} : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A},\ \theta_{\rho'}(\mathbf{x},\mathbf{y}) = \mathbf{x}^*\rho'\mathbf{y} = -i(x_1^*y_2 - x_2^*y_1) = 2\ \mathrm{Im}\ (x_1^*y_2), \tag{34}$$

its unitary group

$$\mathcal{U}_{\rho'} = \{a \in G_{M_2} : \rho' a^* \rho' = a^{-1}\}, \tag{35}$$

and the hyperboloid

$$\mathcal{K}_{\rho'} = \{\mathbf{x} \in \mathcal{A}^2 : \mathbf{x}^* \rho' \mathbf{x} = 1, x_1 \in G_{\mathcal{A}}\}. \tag{36}$$

Consider also the unitary matrix in M_2

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{37}$$

Proposition 16 *The matrix U has the following properties,*

1. *it intertwines ρ and ρ': $\rho' = U\rho U^*$;*
2. *it conjugates the unitary groups: $\mathcal{U}_{\rho'} = U\mathcal{U}_\rho U^*$;*
3. *it maps one hyperboloid onto the other: $\mathcal{K}_{\rho'} = U\mathcal{K}_\rho$.*

Proof The proof is straightforward. Note that for $\mathbf{x} \in \mathcal{K}_\rho$, it holds that $(U\mathbf{x})_1 \in G_{\mathcal{A}}$: in fact, $(U\mathbf{x})_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) = \frac{1}{\sqrt{2}}(1 + x_2 x_1^{-1})x_1$, and $1 + x_2 x_1^{-1} \in G_{\mathcal{A}}$ because $x_2 x_1^{-1} \in \mathcal{D}$ (see Proposition 9).

As in the case of ρ, the group $\mathcal{U}_{\rho'}$ acts on $\mathcal{K}_{\rho'}$ by left multiplication. The action is transitive because $\mathcal{U}_\rho$ acts transitively on $\mathcal{K}_\rho$. □

Proposition 17 *1. If $\mathbf{x} \in \mathcal{K}_{\rho'}$ then $x_2 x_1^{-1} \in \mathcal{H}$.*

2. The map

$$\varphi_H : \mathcal{K}_{\rho'} \to \mathcal{H}\ ,\quad \varphi_H(\mathbf{x}) = x_2 x_1^{-1} \tag{38}$$

is a submersion with a global cross section

$$\sigma_H : \mathcal{H} \to \mathcal{K}_{\rho'}\ ,\quad \sigma_H(h) = \begin{pmatrix} 1 \\ h \end{pmatrix} (2\ Im\ h)^{-1/2}. \tag{39}$$

Proof If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_{\rho'}$ then

$$\text{Im}\,(x_2 x_1^{-1}) = \frac{1}{2i}(x_2 x_1^{-1} - (x_1^*)^{-1} x_2^*) = (x_1^*)^{-1}\{\frac{1}{2i}(x_1^* x_2 - x_2^* x_2)\}x_1^{-1} = (x_1^*)^{-1}\,\text{Im}\,(x_1^* x_2) x_1^{-1}$$

$$= \frac{1}{2}(x_1 x_1^*)^{-1} \in G^+$$

because $1 = \mathbf{x}^* \rho' \mathbf{x} = 2\,\text{Im}\,(x_1^* x_2)$ by (34) and (36). Then φ_H is a well defined smooth map $\mathcal{K}_{\rho'} \to \mathcal{H}$. The fact that σ_H is a cross section is clear. □

As in the case of ρ, we define

$$\mathcal{Q}_{\rho'} = \{q \in \mathcal{Q} : (2q-1)\rho' \in G^+_{M_2}\} \tag{40}$$

and we get analogous results and properties. In particular, if $\mathbf{x} \in \mathcal{K}_{\rho'}$ and $p_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*\rho'$, then $p_{\mathbf{x}} \in \mathcal{Q}_{\rho'}$ and these idempotents exhaust $\mathcal{Q}_{\rho'}$:

$$\mathcal{Q}_{\rho'} = \{\mathbf{x}\mathbf{x}^*\rho' : \mathbf{x} \in \mathcal{K}_{\rho'}\}.$$

We get an analogous commutative diagram

$$\begin{array}{ccc} & \mathcal{K}_{\rho'} & \\ {\scriptstyle \varphi_{\rho'}}\swarrow & & \searrow{\scriptstyle \varphi_H} \\ \mathcal{Q}_{\rho'} & \xrightarrow{\alpha_H} & \mathcal{H}, \end{array} \tag{41}$$

where $\varphi_{\rho'}(\mathbf{x}) = \mathbf{x}\mathbf{x}^*\rho'$ and $\alpha_H(p_{\mathbf{x}}) = x_2 x_1^{-1}$ is a well defined diffeomorphism.

The group $\mathcal{U}_{\rho'}$ acts on $\mathcal{H}$ (as $\mathcal{U}_\rho$ acts on $\mathcal{D}$) as follows. First, note that, if $a \in \mathcal{U}_{\rho'}$ and $h \in \mathcal{H}$ then

$$\mathbf{x}_h = \begin{pmatrix} 1 \\ h \end{pmatrix} (2 \operatorname{Im} h)^{-1/2} \in \mathcal{K}_{\rho'} \tag{42}$$

and also $a\mathbf{x}_h \in \mathcal{K}_{\rho'}$ by Proposition 16.

Definition 4 For $a \in \mathcal{U}_{\rho'}$ and $h \in \mathcal{H}$ put

$$a \cdot h = \varphi_H(a\mathbf{x}_h) = (a\mathbf{x}_h)_2 (a\mathbf{x}_h)_1^{-1} = (a_{21} + a_{22}h)(a_{11} + a_{12}h)^{-1}.$$

It is a routine calculation to prove that this action is well defined and that all maps in the diagram (41) are $\mathcal{U}_{\rho'}$-equivariant. The action of $\mathcal{U}_{\rho'}$ on $\mathcal{H}$ is transitive because the unitary U implements an isomorphism between diagrams (28) and (41). However, in the next section we shall prove this result without using the unitary U.

As in the case of $\mathcal{D}$, $\mathcal{H}$ is a reductive homogeneous space of $\mathcal{U}_{\rho'}$ and we get that $\mathcal{H}$ and $\mathcal{Q}_{\rho'}$ are isomorphic (as reductive homogeneous spaces). Combining diagrams (28) and (41) by means of U, we get a bigger diagram

$$\begin{array}{ccccccc} & \mathcal{K}_\rho & & \xrightarrow{U} & & \mathcal{K}_{\rho'} & \\ {\scriptstyle \varphi_\rho}\swarrow & & \searrow{\scriptstyle \varphi_D} & & {\scriptstyle \varphi_{\rho'}}\swarrow & & \searrow{\scriptstyle \varphi_H}. \\ \mathcal{D} & \xrightarrow{\alpha_D^{-1}} & \mathcal{Q}_\rho & \xrightarrow{Ad(U)} & \mathcal{Q}_{\rho'} & \xrightarrow{\alpha_H} & \mathcal{H} \end{array} \tag{43}$$

Here $Ad(U) : \mathcal{Q}_\rho \to \mathcal{Q}_{\rho'}$ denotes the isomomorphism $Ad(U)q = UqU^*$.

Remark 8 The composition $\alpha_H \circ Ad(U) \circ \alpha_D^{-1}$ provides the (linear fractional) diffeomorphism

$$\Gamma : \mathcal{D} \to \mathcal{H}\,, \quad \Gamma(z) = i(1+z)(1-z)^{-1}$$

with inverse

$$\Gamma^{-1} : \mathcal{H} \to \mathcal{D}\,, \quad \Gamma^{-1}(h) = (1+ih)(1-ih)^{-1}.$$

2.9 The Borel Subgroup of $\mathcal{U}_{\rho'}$

We have proven several results about the relations between the groups $\mathcal{U}_\rho$ and the manifolds $\mathcal{K}_\rho, \mathcal{Q}_\rho$ and $\mathcal{D}$. The unitary matrix U allowed us to translate these relations to the analogous spaces in the case of ρ'. We have chosen to proceed in this order (first ρ, then ρ') because many of the properties reviewed had easier and more natural proofs in the former realm. In this section we introduce a subgroup of $\mathcal{U}_{\rho'}$ which is easier to understand, and, more important, it admits a clear geometric interpretation.

Definition 5 The *Borel subgroup* of $\mathcal{U}_{\rho'}$ is

$$\mathcal{B}' := \{a = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{U}_{\rho'}\}.$$

Proposition 18 *The subgroup $\mathcal{B}'$ coincides with*

$$\{a = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} : g \in G_{\mathcal{A}}, x \in \mathcal{A}_h\}.$$

Proof Observe first that if $a = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{U}_{\rho'}$ then $a_{11}, a_{22} \in G_{\mathcal{A}}$, and

$$a^{-1} = \begin{pmatrix} a_{11}^{-1} & 0 \\ -a_{22}^{-1}a_{21}a_{11}^{-1} & a_{22}^{-1} \end{pmatrix}.$$

Then, from $\rho' a^* \rho' = a^{-1}$ we get that $a_{22}^* = a_{11}^{-1}$, $a_{21}^* = a_{11}^* a_{21} a_{22}^* = a_{11}^* a_{21} a_{11}^{-1}$ and $x = a_{21}^* a_{11} \in \mathcal{A}_h$. If we put $g = a_{11}$, then $(g^*)^{-1}x = (a_{11}^*)^{-1}a_{21}^* a_{11} = a_{21}$ and $(g^*)^{-1} = (a_{11}^*)^{-1} = a_{22}$. □

Remark 9 Observe that, for $a = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} \in \mathcal{B}'$ and $h \in \mathcal{H}$, it holds that

$$a \cdot h = (a \begin{pmatrix} 1 \\ h \end{pmatrix})_2 (a \begin{pmatrix} 1 \\ h \end{pmatrix})_1^{-1} = (g^*)^{-1}(x+h)g^{-1} = L_g(a+h).$$

That is, the action of $\mathcal{B}'$ on h is the affine translation by $x^* = x$ (which remains in $\mathcal{H}$: Im $(x+h)$= Im $(h) > 0$), followed by the action L_g of $G_{\mathcal{A}}$ extended to $\mathcal{H}$: $L_g k = (g^*)^{-1} k g^{-1}$.

Significantly, there is no loss of geometric information if we restrict the action to the subgroup $\mathcal{B}'$:

Proposition 19 *$\mathcal{B}'$ acts transitively on $\mathcal{H}$.*

Proof It suffices to prove that, for any $h \in \mathcal{H}$, there exists $a \in \mathcal{B}'$ such that $a \cdot i = h$. Put $h = h_0 + ih_1$ where $h_0 = \operatorname{Re} h$ and $h_1 = \operatorname{Im} h$. If $a = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix}$, then $a \cdot i = (g^*)^{-1}(x+i)g^{-1}$. So, we need to find g, x such that

$$h_0 + ih_1 = (g^*)^{-1}xg^{-1} + i(g^*)^1 g^{-1},$$

i.e., $h_0 = (g^*)^{-1}xg^{-1}$ and $h_1 = (g^*)^1 g^{-1}$. Since $h_1 \in G^+$, then $g = h_1^{1/2}$ verifies the second relation. Pick $x = h_1^{-1/2}h_0h^{-1/2} \in \mathcal{A}_h$ and the second relation holds. □

Proposition 20 *The action of $\mathcal{B}'$ on $\mathcal{K}_{\rho'}$ is free. So in particular, $\mathcal{B}'$ is diffeomorphic to $\mathcal{K}_{\rho'}$.*

Proof For the first assertion, it suffices to prove that for every $\mathbf{x} \in \mathcal{K}_{\rho'}$ there is a unique $b \in \mathcal{B}'$ such that $b\mathbf{x}_i = \mathbf{x}$, if $\mathbf{x}_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, with the notation of (42). So, we look for $g \in G_{\mathcal{A}}$ and $x \in \mathcal{A}_h$ such that

$$b\mathbf{x}_i = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix}\mathbf{x}_i = \mathbf{x}.$$

Explicitly, we need to solve

$$\frac{1}{\sqrt{2}}gx_1\ , \quad \frac{1}{\sqrt{2}}(g^*)^{-1}(x+i) = x_2.$$

The unique solution is

$$g = \sqrt{2}x_1\ , \quad x = 2x_1^*x_2 - i.$$

Note that $x^* = x$, because $\mathbf{x} \in \mathcal{K}_{\rho'}$ means that $x_1^*x_2 - x_2^*x_1 = i$.

Thus, we also get the diffeomorphism

$$\mathcal{K}_{\rho'} \to \mathcal{B}' \quad \mathbf{x} \mapsto \begin{pmatrix} \sqrt{2}x_1 & 0 \\ \frac{1}{\sqrt{2}}(x_1^*)^{-1}(x_1^*x_2 + x_2^*x_1) & \frac{1}{\sqrt{2}}(x_1^*)^{-1} \end{pmatrix} \tag{44}$$

with inverse

$$\mathcal{B}' \to \mathcal{K}_{\rho'}\ , \quad \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} \mapsto \begin{pmatrix} g \\ (g^*)^{-1}(x+i) \end{pmatrix}.$$

□

Remark 10 The diffeomorphism $\mathcal{K}_{\rho'} \simeq \mathcal{B}'$ proved above defines a group structure on $\mathcal{K}_{\rho'}$. On the other hand, as in (22), we have the diagram

$$\begin{array}{ccc} \mathcal{K}_{\rho'} & \hookrightarrow & \{(a, \mathbf{x}) :\in \mathcal{Q}_{\rho'} \times \mathcal{A}^2 : \mathbf{x} \in R(q)\} \\ & \searrow & \downarrow \\ & & \mathcal{Q}_{\rho'}, \end{array} \tag{45}$$

which proves that the map $\mathcal{K}_{\rho'} \to \mathcal{Q}_{\rho'}$, $\mathbf{x} \mapsto p_{\mathbf{x}}$ is a principal fiber bundle with structure group $\mathcal{U}_{\mathcal{A}}$, associated to the tautological bundle on $\mathcal{Q}_{\rho'}$, cf. [41]. This configuration is similar to that of the classical *Hopf fibration*

$$\begin{array}{ccc} \mathbf{S^1} & \hookrightarrow & \mathbf{S^3} \subset \mathbb{C}^2 \\ & & \downarrow \\ & & \mathbf{S^2} = \mathbb{CP}^1, \end{array}$$

where $\mathbf{S^3} \to \mathbf{S^2}$ is the principal $\mathbf{S^1}$-bundle associated to the tautological bundle on $\mathbb{CP}^1$. Moreover, $\mathbf{S^3}$ can be identified with the group Spin (3), which is isomorphic to the group of unit quaternions.

In order to understand the structure of $\mathcal{U}_{\rho'}$, it will be also useful to consider the subgroup $\mathcal{T} \subset \mathcal{U}_{\rho'}$,

$$\mathcal{T} := \{\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} : \tau \in \mathcal{A}_h\}. \tag{46}$$

As we shall see, locally, $\mathcal{B}'$ and $\mathcal{T}$ decompose $\mathcal{U}_{\rho'}$.

Proposition 21 *$\mathcal{B}'$ and $\mathcal{T}$ are Banach-Lie subgroups of $\mathcal{U}_{\rho'}$, and complemented submanifolds of M_2. They generate an open and closed subgroup of $\mathcal{U}_{\rho'}$, which contains the connected component of the identity.*

Proof First note that the diagonal entries of elements in $\mathcal{B}'$ must be invertible elements in $\mathcal{A}$. Elementary matrix computations show that $b \in \mathcal{B}$ if and only if it is of the form

$$b = \begin{pmatrix} g & 0 \\ x & (g^*)^{-1} \end{pmatrix},$$

with $g^* x g^{-1} = x^*$, or, equivalently, $g^* x$ selfadjoint. Thus, $\mathcal{B}'$ can be parametrized by

$$\mathbb{B} = \{(g, x) : g \in G_{\mathcal{A}} \text{ and } g^* x \in \mathcal{A}_h\}.$$

This set $\mathbb{B}$ is globally diffeomorphic to the set

$$G_{\mathcal{A}} \times \mathcal{A}_h = \{(a, y) : a \in G_{\mathcal{A}}, y = y^*\}.$$

which is a complemented submanifold of $\mathcal{A}^2$. The diffeomorphism and its inverse are given by $(g, x) \mapsto (g, g^* x)$ and $(a, y) \mapsto (a, (a^*)^{-1} y)$. Thus, $\mathcal{B}'$ is globally

diffeomorphic to a submanifold of $\mathcal{A}^2$. Moreover, this diffeomorphism extends to an open subset of M_2:

$$\{\begin{pmatrix} a & c \\ y+z & d \end{pmatrix} : (a,y) \in G \times \mathcal{A}_h,\ z,c,d \in \mathcal{A},\ z^* = -z\} \to M_2,$$

$$\begin{pmatrix} a & c \\ y+z & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ (a^*)^{-1}y+z & (a^*)^{-1}+d \end{pmatrix}$$

which maps $G_{\mathcal{A}} \times \mathcal{A}_h$ onto $\mathcal{B}'$.

The Banach-Lie algebra $a_{\mathcal{B}'}$ of $\mathcal{B}'$ can be computed using this parametrization. If $g(t) \in G_{\mathcal{A}}$ and $x(t) \in \mathcal{A}$ are smooth curves such that $g^*(t)x(t)$ is selfadjoint, $g(0) = 1$, $\dot{g}(0) = y_{11}$, $x(0) = 0$ and $\dot{x}(0) = y_{21}$, then

$$b(t) = \begin{pmatrix} g(t) & 0 \\ x(t) & (g^*(t))^{-1} \end{pmatrix}$$

is a smooth curve in $\mathcal{B}'$ with $b(0) = 1$ and $\dot{b}(0) = Y$,

$$Y = \begin{pmatrix} y_{11} & 0 \\ y_{21} & -y_{11}^* \end{pmatrix},$$

where $\frac{d}{dt}\{g^*(t)x(t)\} = \dot{g^*}(t)x(t) + g^*(t)\dot{x}(t) \in \mathcal{A}_h$; in particular, at $t = 0$, this implies that y_{21} is selfadjoint. Thus, the Banach-Lie algebra $a_{\mathcal{B}'}$ of $\mathcal{B}'$ is

$$a_{\mathcal{B}'} = \{Y = \begin{pmatrix} y_{11} & 0 \\ y_{21} & -y_{11}^* \end{pmatrix} : y_{21}^* = y_{21}\}.$$

The subgroup $\mathcal{T}$ is parametrized by $\mathcal{A}_h$, and can be proved to be a submanifold of M_2 in a similar (simpler) fashion. Also, it is apparent that the Banach-Lie algebra $a_{\mathcal{T}}$ of $\mathcal{T}$ is

$$a_{\mathcal{T}} = \{Z = \begin{pmatrix} 0 & z_{12} \\ 0 & 0 \end{pmatrix} : z_{12}^* = z_{12}\}.$$

We claim that the subgroups $\mathcal{B}$ and $\mathcal{T}$ generate a closed and open subgroup of $\mathcal{U}_{\rho'}$. To prove this, note that the Banach-Lie algebras of these groups are in direct sum, and its sum is the Banach-Lie algebra $\mathfrak{U}_{\rho'}$ of $\mathcal{U}_{\rho'}$:

$$\mathfrak{U}_{\rho'} = a_{\mathcal{B}} \oplus a_{\mathcal{T}}.$$

It follows that there is a neighbourhood $\mathcal{W}$ of 1 in $\mathcal{U}_{\rho'}$ where any element is the product of elements in $\mathcal{B}'$ and $\mathcal{T}$. Let $a_0 \in \mathcal{U}_{\rho'}$ which is a product of elements in $\mathcal{B}'$ and $\mathcal{T}$ in a unique fashion. Then

$$\mathcal{W}_0 = \{a : a_0^{-1}a \in \mathcal{W}\}$$

is an open neighbourhood of a_0 in $\mathcal{U}_{\rho'}$. It follows that $\mathcal{B}'$ and $\mathcal{T}$ generate an open subgroup of $\mathcal{U}_{\rho'}$. The relation

$$a \sim b \text{ if and only if } a^{-1}b \in \mathcal{B}'\mathcal{T}$$

is an equivalence relation. It follows that this subgroup is a union of connected components of $\mathcal{U}_{\rho'}$, containing the connected component of the identity. □

Remark 11 If $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{U}_{\rho'}$ is connected. Since $\mathcal{U}^+_{\rho'}$ is clearly connected (in fact, contractible), one needs to show that the unitary part $\mathbf{D}_{\rho'} = \mathcal{U}_{\rho'} \cap \mathcal{U}_{M_2}$ of $\mathcal{U}_{\rho'}$ is connected. Note that $\mathbf{D}_{\rho'} = \mathcal{U}_{M_2} \cap \{J\}'$. Since J is anti-Hermitian, it follows that $\{J\}' \subset M_2$ is a von Neumann algebra, and therefore $\mathcal{U}_{M_2} \cap \{J\}'$ is the unitary group of a von Neumann algebra, thus connected.

Theorem 7 *The unitary part $\mathbf{D}_{\rho'}$ of $\mathcal{U}_{\rho'}$ is isomorphic to $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{A}}$. The group $\Pi_0(\mathcal{U}_{\rho'})$ of connected components of $\mathcal{U}_{\rho'}$, is isomorphic to $\Pi_0(\mathcal{U}_{\mathcal{A}}) \times \Pi_0(\mathcal{U}_{\mathcal{A}})$.*

Proof Consider the map

$$\{J\}' \to \mathcal{A}, \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + ib.$$

This map is an injective C*-homomorphism; thus, its restriction to $\mathcal{U}_{M_2} \cap \{J\}'$

$$\Gamma : \mathcal{U}_{M_2} \cap \{J\}' \to \mathcal{U}_{\mathcal{A}}$$

is a group homomorphism. It is a retraction: the map $u \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ is a cross section for Γ and a group homomorphism. By straightforward computations, the kernel of Γ consist of matrices

$$\begin{pmatrix} a & i(a-1) \\ -i(a-1) & a \end{pmatrix}$$

with $a^*a = aa^* = \frac{1}{2}(a + a^*)$. Then a is a normal element, which is a zero of the continuous function $f(z) = |z|^2 - Re(z)$. Therefore, the spectrum of a is contained in the zero set of f, namely $\{z \in \mathbb{C} : |z - \frac{1}{2}| = \frac{1}{2}\}$. The map $z \mapsto 2z - 1$ sends this circle to the unit circle, and since it is a polynomial map, it sends elements a as above onto normal elements with spectrum in the unit circle, i.e., unitary elements of $\mathcal{A}$. Conversely, if $u \in \mathcal{U}_{\mathcal{A}}$, elementary computations show that $a = \frac{1}{2}(u + 1)$ satisfies $a^*a = aa^* = \frac{1}{2}(a + a^*)$. Moreover, it is easy to verify that the map

$$\ker \Gamma \to \mathcal{U}_{\mathcal{A}}, \quad \begin{pmatrix} a & i(a-1) \\ -i(a-1) & a \end{pmatrix} \mapsto 2a - 1$$

is a group homomorphism, thus a bicontinuous isomorphism.

Therefore, since Γ splits, one has that (by means of an explicit isomorphism)

$$\mathcal{U}_{M_2} \cap \{J\}' \xrightarrow{\simeq} \mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{A}}.$$

The polar decomposition induces the isomorphism between $\Pi_0(\mathcal{U}_{\rho'})$ and $\Pi_0(\mathcal{U}_{M_2} \cap \{J\}')$, because the positive part $\mathcal{U}_{\rho'}^+$ is contractible. □

Corollary 8 *$\mathcal{B}'$ and $\mathcal{T}$ generate $\mathcal{U}_{\rho'}$ if and only if $\mathcal{U}_{\mathcal{A}}$ is connected.*

Remark 12 Note that if $c \in \mathcal{T}$, then $c^* \in \mathcal{B}'$. In particular, $\mathcal{B}'$ and $(\mathcal{B}')^* = \{b^* : b \in \mathcal{B}'\}$ generate a subgroup $\mathcal{B}' \vee (\mathcal{B}')^*$ of $\mathcal{U}_{\rho'}$ which contains the connected component of the identity.

The matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an element of $\mathcal{U}_{\rho'}$. Notice that $\mathcal{B}'$ and J generate the subgroup $\mathcal{B}' \vee J = \mathcal{B}' \vee (\mathcal{B}')^*$. Indeed, if $b = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} \in \mathcal{B}'$ $(x^* = x)$, then

$$JbJ^{-1} = \begin{pmatrix} (g^*)^{-1} & -(g^*)^{-1} \\ 0 & g \end{pmatrix},$$

which is the adjoint of $\begin{pmatrix} g^{-1} & 0 \\ -xg^{-1} & g^* \end{pmatrix} = \begin{pmatrix} h & 0 \\ (h^*)^{-1}y & (h^*)^{-1} \end{pmatrix}$, an element of $\mathcal{B}'$; here $h = g^{-1}$ and $y = -h^*xh \in \mathcal{A}_h$.

Observe that J acts as a symmetry of $\mathcal{H}$ in the sense that it is an involutive isometry of the homogeneous reductive space $\mathcal{H}$ and inverts geodesics through $i \in \mathcal{H}$. Also we have a similar symmetry with respect to any point in $\mathcal{H}$ (by means of the action of the group $\mathcal{U}_{\rho'}$). For this reason we can consider $\mathcal{H}$ as a symmetric space as in the classical setting [42].

2.10 The Covariant Derivative in $\mathcal{H}$

In this section we compute explicitly the covariant derivative induced by the reductive structure (see for instance [36]). As in the case of $\mathcal{U}_\rho$, the Banach-Lie algebra $\mathfrak{u}_{\rho'}$ of $\mathcal{U}_{\rho'}$ decomposes as

$$\mathfrak{u}_{\rho'} = \mathfrak{u}_{0,\rho'} \oplus \mathfrak{u}_{1,\rho'} = \{\beta \in M_2(\mathcal{A})_{as} : \beta J = J\beta\} \oplus \{\gamma \in M_2(\mathcal{A})_s : \gamma J = -J\gamma\},$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, elements $X \in \mathfrak{u}_{\rho'}$ are of the form $X = X_0 + X_h$,

$$X = \begin{pmatrix} x_{11} & x_{12} \\ -x_{12} & -x_{11} \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

with $x_{11}^* = -x_{11}$, and all other entries selfadjoint. The left hand subspace $\mathfrak{u}_{0,\rho'}$ is the Banach-Lie algebra of the isotropy group of the action at the element $i \in \mathcal{H}$. The right hand subspace $\mathfrak{u}_{1,\rho'}$ is the *horizontal* space at this point.

The computation of the covariant derivative will be done in several steps.

Step 1. First we compute the tangent map of $\pi_i : \mathcal{U}_{\rho'} \to \mathcal{H}, \pi_i(g) = g \cdot i$, at the identity $1 \in \mathcal{U}_{\rho'}$. Recall that $\pi_i(g) = (g_{22}i + g_{21})(g_{12}i + g_{11})^{-1}$. Then, differentiating at 1, we get

$$(T\pi_i)_1(\gamma) = \gamma_{21} + \gamma_{12} + i(\gamma_{22} - \gamma_{11}).$$

If γ is horizontal, then $(T\pi_i)_1(\gamma) = 2\gamma_{12} - 2i\gamma_{11}$.

Theorem 8 *The* 1*-form of the reductive connection at* $i \in \mathcal{H}$ *is*

$$\varsigma_i(\zeta) = \frac{1}{2}\begin{pmatrix} -\Upsilon & \chi \\ \chi & \Upsilon \end{pmatrix}, \text{ if } \zeta = \chi + i\Upsilon.$$

Step 2. For any given $h = x + iy$, one can find an element $b \in \mathcal{B}' \subset \mathcal{U}_{\rho'}$ (the Borel subgroup of $\mathcal{U}_{\rho'}$) such that $b \cdot i = h$. For instance,

$$b = \begin{pmatrix} y^{-1/2} & 0 \\ xy^{-1/2} & y^{1/2} \end{pmatrix} \text{ with inverse } b^{-1} = \begin{pmatrix} y^{1/2} & 0 \\ -y^{-1/2}x & y^{-1/2} \end{pmatrix}.$$

Straightforward computations show that $b \cdot i = h$ (and $b^{-1} \cdot h = i$).

Step 3 Let us compute now the tangent map of the action of $g \in \mathcal{U}_{\rho'}$ on tangent vectors of $\mathcal{H}$. If $\frac{d}{dt}h = \zeta$, then

$$\frac{d}{dt}(g \cdot h) = (g_{22} - wg_{12})\zeta(g_{12}h + g_{11})^{-1},$$

where $w = g \cdot h = (g_{22}h + g_{21})(g_{12}h - g_{11})^{-1}$. Using the transformation in **Step 2**, we can carry a tangent vector ζ at h to the tangent vector $y^{-1/2}\zeta y^{-1/2}$ at i. If $\zeta = \chi + i\Upsilon$,

$$\varsigma_i(y^{-1/2}\zeta y^{-1/2}) = \frac{1}{2}y^{-1/2}\begin{pmatrix} -\Upsilon & \chi \\ \chi & \Upsilon \end{pmatrix}y^{-1/2}.$$

Step 4 To obtain $\varsigma_h(\zeta)$, we use the inner automorphism Ad_b:

$$\begin{pmatrix} y^{-1/2} & 0 \\ xy^{-1/2} & y^{-1/2} \end{pmatrix}\{\frac{1}{2}y^{-1/2}\begin{pmatrix} -\Upsilon & \chi \\ \chi & \Upsilon \end{pmatrix}y^{-1/2}\}\begin{pmatrix} y^{1/2} & 0 \\ -y^{-1/2}x & y^{-1/2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -y^{-1}\chi y^{-1}x & y^{-1}\chi y^{-1} \\ \chi - xy^{-1}\chi y^{-1}x & xy^{-1}\chi y^{-1} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -y^{-1}\Upsilon & 0 \\ -xy^{-1}\Upsilon - \Upsilon y^{-1}x & \Upsilon y^{-1} \end{pmatrix}.$$

Step 5 Next, we compute the covariant derivative of the reductive connection in $\mathcal{H}$, at the point $i \in \mathcal{H}$. Consider the curve $h(t) = x(t) + iy(t)$ in $\mathcal{H}$, with $z(0) = i$, and the vector field $\zeta = \chi + i\Upsilon$ defined on a neighbourhood of i. Denote by $\frac{D\zeta}{dt}|_{t=0}$ the covariant derivative at $i \in \mathcal{H}$. According to [22], one has that

$$\varsigma_i(\frac{D\zeta}{dt}|_{t=0}) = \frac{d}{dt}\varsigma_{h(t)}Z(h(t))|_{t=0} + [\varsigma_i(Z(i)), \varsigma_i(\frac{d}{dt}h(t)|_{t=0}].$$

To lighten the notation, we shall denote by $\varsigma(\zeta)'$, h' the usual derivatives at $t = 0$. So the formula above reads

$$\varsigma_i(\frac{D\zeta}{dt}|_{t=0}) = \varsigma(\zeta)' + [\varsigma_i(Z(i)), \varsigma_i(h')].$$

Then

$$\varsigma(\zeta)' = \frac{1}{2}\begin{pmatrix} -\Upsilon' & \chi' \\ \chi' & \Upsilon' \end{pmatrix} + \frac{1}{2}\begin{pmatrix} y'\Upsilon - \chi x' & -y'\chi - \chi y' \\ -x'\chi - \Upsilon x' & x'\chi - \Upsilon y' \end{pmatrix}$$

Let us write the sum of right hand matrix above with the bracket $[\varsigma_i(Z(i)), \varsigma_i(h')]$. After strenuous but elementary computations one gets

$$\frac{1}{4}\begin{pmatrix} \Upsilon y' - \chi x' & -\Upsilon x' - \chi y' \\ -\chi y' - \Upsilon x' & \chi x' - \Upsilon y' \end{pmatrix} + \frac{1}{4}\begin{pmatrix} y'\Upsilon - x'\chi & -y'\chi - x'\Upsilon \\ -x'\Upsilon - y'\chi & x'\chi - y'\Upsilon \end{pmatrix}.$$

Now we must apply ς_i^{-1}, or else realize that the above term is the value of ς_i at

$$-\,Re(x'\Upsilon + y'\chi) + i\,Re(x'\chi - y'\Upsilon).$$

Therefore,

$$\frac{d\zeta}{dt}|_{t=0} = \chi' + i\Upsilon' + \{-Re(x'\Upsilon + y'\chi) + i\,Re(x'\chi - y'\Upsilon)\}.$$

Step 6 Let $h_0 = x_0 + iy_0 \in \mathcal{H}$, and $h(t) = x(t) + iy(t)$ be a smooth curve in $\mathcal{H}$ with $h(0) = h_0$. Let $\zeta = \chi + i\Upsilon$ be a smooth vector field defined on a neighbourhood of h_0. Let us compute $\frac{D\zeta}{dt}|_{t=0}$. To do this, we shall use the invariance of the connection under the action of the group $\mathcal{U}_{\rho'}$, and the fact that we know this formula in the case $h_0 = i$. As seen in Step 2, the (explicit) element b defined there, performs $b^{-1} \cdot h_0 = i$. Thus we can carry the data to the point i, perform the covariant derivative, and translate it back to h_0 with the action (note for instance, that $y_0^{1/2} Re(u) y_0^{1/2} = Re(y_0^{1/2} u y_0^{1/2})$, and so forth). We get:

$$\frac{D\zeta}{dt} = \zeta' - Re(x'y_0^{-1}\Upsilon + y'y_0^{-1}\chi) + i\,Re(x'y_0^{-1}\chi - y'y_0^{-1}\Upsilon). \qquad (47)$$

Therefore, the geodesics ζ of $\mathcal{H}$, which start in $h_0 = x_0 + iy_0$, are the solutions of $\frac{D\dot{\zeta}}{dt} = 0$, which amounts to

$$\begin{cases} \ddot{\chi} = Re\,(x'y_0^{-1}\dot{\Upsilon} + y'y_0^{-1}\dot{\chi}) \\ \ddot{\Upsilon} = Re\,(x'y_0^{-1}\dot{\chi} - y'y_0^{-1}\dot{\Upsilon}) \end{cases}.$$

3 Metric Structure

By "Finsler metric" or "Finsler structure", we mean a continuous distribution of norms in the tangent spaces as in Sect. 2.1 (cf. [27]).

3.1 Finsler Metrics

In this section we define Finsler structures (in the sense given to this notion in Sect. 2.1) in $\mathcal{Q}_\rho$ and $\mathcal{D}$ which are invariant with respect to the action of $\mathcal{U}_\rho$. The case of $\mathcal{Q}_{\rho'}$ and $\mathcal{H}$ as homogeneous metric spaces of $\mathcal{U}_{\rho'}$ can be dealt similarly.

3.1.1 Finsler Metric in $\mathcal{Q}_\rho$

The space $\mathcal{Q}_\rho$ has a Finsler metric, which is invariant under the action of $\mathcal{U}_\rho$. It is defined as follows:

1. If $\mathbf{X} \in (T\mathcal{Q}_\rho)_p$, define $\|\mathbf{X}\|_p = \|\mathbf{X}\|$.
2. If $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$, then $a^{-1}\mathbf{X}a \in (T\mathcal{Q}_\rho)_p$ for every $a \in \mathcal{U}_\rho$ such that $a \cdot p = q$. We define $\|\mathbf{X}\|_q = \|a^{-1}\mathbf{X}a\|$.

Note that $\|a^{-1}\mathbf{X}a\| = \|b^{-1}\mathbf{X}b\|$ if $a, b \in \mathcal{U}_\rho$ verify that $a \cdot p = b \cdot p = q$. Indeed, in this case $b = aw$ for $w \in \mathcal{I}_p$; in particular, as we saw earlier, w is a unitary element in M_2. Thus

$$\|b^{-1}\mathbf{X}b\| = \|w^* a^{-1}\mathbf{X}aw\| = \|a^{-1}\mathbf{X}a\|.$$

Thus the Finsler metric below is well defined:

Definition 6 For $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$,

$$\|\mathbf{X}\| = \|a^{-1}\mathbf{X}a\|\,|,$$

for any $a \in \mathcal{U}_\rho$ such that $a \cdot p = q$

The invariance of this metric under the action of $\mathcal{U}_\rho$ is clear. Since the action of $\mathcal{U}_\rho$ on $\mathcal{Q}_\rho$ is the restriction of the linear action on the whole algebra M_2 (by inner conjugation), the action induced on tangent vectors is also given by inner conjugation, i.e. if $c \in \mathcal{U}_\rho$, then $(TL_c)_{L_c q}(\mathbf{X}) = c\mathbf{X}c^{-1}$. Thus,

$$\|c\mathbf{X}c^{-1}\|_{cqc^{-1}} = \|a^{-1}c\mathbf{X}c^{-1}a\| = \|(c^{-1}a)^{-1}\mathbf{X}c^{-1}a\|$$

for any $a \in \mathcal{U}_\rho$ such that $apa^{-1} = cqc^{-1}$. Then $c^{-1}a \cdot p = q$, and therefore this last norm equals $\|\mathbf{X}\|_q$ by definition. That is,

$$\|c\mathbf{X}c^{-1}\|_{cqc^{-1}} = \|\mathbf{X}\|_q\ , \quad \text{for all } q \in \mathcal{Q}_\rho,\ \mathbf{X} \in (T\mathcal{Q}_\rho)_q \text{ and } c \in \mathcal{U}_\rho. \tag{48}$$

3.1.2 Finsler Metric in $\mathcal{D}$

We introduce a Finsler metric on $\mathcal{D}$, which will be shown to be invariant under the action of $\mathcal{U}_\rho$. In other words, for each $a \in \mathcal{U}_\rho$, if $L_a : \mathcal{D} \to \mathcal{D}$ denotes the translation $L_a z = a \cdot z$, then $(TL_a)_z : (T\mathcal{D})_z \to (T\mathcal{D})_{a\cdot z}$ will be an isometry.

Definition 7 1. For $x \in (T\mathcal{D})_0 \simeq \mathcal{A}$ we put

$$\|x\|_0 := \|x\|, \tag{49}$$

the usual norm of $\mathcal{A}$.

2. If $x \in (T\mathcal{D})_z$ where $z = a \cdot 0$, then $(TL_{a^{-1}})_z x \in (T\mathcal{D})_0$ and we put

$$\|x\|_z := \|(TL_{a^{-1}})_z x\|_0 = \|(TL_{a^{-1}})_z x\|. \tag{50}$$

Remark 13 We must check that this is a good definition, i.e.. that if $a \cdot 0 = b \cdot 0 = z$ for $a, b \in \mathcal{U}_\rho$, then

$$\|(TL_{a^{-1}})_z x\| = \|(TL_{b^{-1}})_z x\|$$

for any $x \in (T\mathcal{D})_z$. But if $a \cdot 0 = b \cdot 0$ then

$$b^{-1}a = w \in \mathcal{I}_0 = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \in \mathcal{U}_\mathcal{A}\}.$$

Note that the action of the unitary element $w \in \mathcal{I}_0$ on $\mathcal{D}$ is given by

$$w \cdot z = vzu^*,$$

which is the restriction to $\mathcal{D}$ of the linear and isometric isomorphism $T_{v,u} : \mathcal{A} \to \mathcal{A}$, $T_{v,u}x = vxu^*$. Note that

$$L_{b^{-1}} = L_{wa^{-1}} = L_w \circ L_{a^{-1}} = T_{v,u} \circ L_{a^{-1}},$$

and since $T_{v,u}$ is a linear map, by differentiation, we get

$$(TL_{b^{-1}})_z = T_{v,u} \circ (TL_{a^{-1}})_z.$$

Finally, since $T_{v,u}$ is isometric,

$$\|(TL_{b^{-1}})_z x\| = \|T_{v,u} \circ (TL_{a^{-1}})_z x\| = \|(TL_{a^{-1}})_z x\|,$$

as claimed.

Next, we prove that the diffeomorphism α_D of (29) is isometric.

Proposition 22 *The diffeomorphism* $\alpha_D : \mathcal{Q}_\rho \to \mathcal{D}$, $\alpha_D(q) = q_{21}q_{11}^{-1}$ *is isometric, i.e.*

$$\|(T\alpha_D)_q x\|_{\alpha_D(q)} = \|x\|_q,$$

for every $x \in (T\mathcal{Q}_\rho)_q$.

Proof Since $\mathcal{U}_\rho$ acts on $\mathcal{D}$ and $\mathcal{Q}_\rho$ and the map α_D is equivariant, it suffices to check this assertion for $q = p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $\alpha_D(p) = 0$. We have that

$$(T\mathcal{Q}_\rho)_p = \{x \in M_2 : \rho x^* \rho = x, xp + px = x\} = \{\begin{pmatrix} 0 & x_{21}^* \\ x_{21} & 0 \end{pmatrix} : x_{21} \in \mathcal{A}\},$$

and

$$(T\alpha_D)_q x = x_{21}q_{11}^{-1} - q_{21}x_{11}\ , \quad x \in (T\mathcal{Q}_\rho)_q.$$

So $(T\alpha_D)_p x = x_{21}$, $x \in (T\mathcal{Q}_\rho)_p$. Thus,

$$\|(T\alpha_D)_p x\|_0 = \|x_{21}\| = \|\begin{pmatrix} 0 & x_{21}^* \\ x_{21} & 0 \end{pmatrix}\| = \|x\|.$$

□

Remark 14 There is a similar result for the isomorphism $\mathcal{Q}_{\rho'} \to \mathcal{H}$.

3.2 Immersions in $G^+_{M_2}$

We have now all geometric tools necessary to profit from our knowledge of the geometry of $\mathcal{Q}_\rho$ and $\mathcal{Q}_{\rho'}$. First, we need the following definition:

Definition 8 If M (resp. M') is a homogeneous space of the Lie-Banach group G (resp. G'), a *morphism* from (G, M) to (G', M') is a smooth map

$$G \times M \to G' \times M'\ , \quad (g, m) \mapsto (\varphi(g), \Phi(m))$$

such that $\varphi : G \to G'$ is a smooth group homomorphism, $\Phi : M \to M'$ is a smooth map and

$$\Phi(g \cdot m) = \varphi(g) \cdot \Phi(m) \tag{51}$$

for all $g \in g, m \in M$.

Examples 9 1. $\mathcal{U}_\rho \times \mathcal{D} \to \mathcal{U}_\rho \times \mathcal{Q}_\rho$, $\varphi(a) = a$, $\Phi(z) = p_{\mathbf{x}_z}$, where as before, $\mathbf{x}_z = \begin{pmatrix} 1 \\ z \end{pmatrix}(1 - z^*z)^{-1/2}$. In this case it is an isomorphism, i.e., the inverse map is also an morphism.

2. $\mathcal{U}_\rho \times \mathcal{Q}_\rho \to G_{M_2} \times G^+_{M_2}$, $\varphi(a) = (a^*)^{-1}$ (i.e., φ is the inclusion $\mathcal{U}_\rho \hookrightarrow G_{M_2}$), $\Phi(q) = (2q-1)\rho$ $(q \in \mathcal{Q}_\rho)$. Indeed, it holds that $\Phi(aqa^{-1}) = a(2q-1)a^*$. Notice that in $im\ \Phi = \mathcal{U}^+_\rho \subset G^+_{M_2}$ (see Proposition 6).
Analogous examples of morphisms are provided by the following versions of 1. and 2.:
3. $\mathcal{U}_{\rho'} \times \mathcal{H} \to \mathcal{U}_{\rho'} \times \mathcal{Q}_{\rho'}$, $\varphi(a) = a$, $\Phi(h) = p_{\mathbf{x}_h}$, where $\mathbf{x}_h = \begin{pmatrix} 1 \\ h \end{pmatrix} (\mathrm{Im}\ h)^{-1/2}$, as in (42).
4. $\mathcal{U}_{\rho'} \times \mathcal{Q}_{\rho'} \to G_{M_2} \times G^+_{M_2}, \varphi(a) = (a^*)^{-1}, \Phi'(q') = (2q'-1)\rho'$. Here also $im\ \Phi' = \mathcal{U}^+_{\rho'} = \mathcal{U}_{\rho'} \cap G^+_{M_2}$.

It is easy to check that the immersions $\Phi : \mathcal{Q}_\rho \to G^+_{M_2}$ and $\Phi' : \mathcal{Q}_{\rho'} \to G^+_{M_2}$ preserve the Finsler metrics.

Therefore, we get:

Theorem 10 *The spaces $\mathcal{D}, \mathcal{H}, \mathcal{Q}_\rho$ and $\mathcal{Q}_{\rho'}$ share with $G^+_{M_2}$ the following properties:*

1. *any two points can be joined by a unique geodesic;*
2. *every geodesic is defined for all $t \in \mathbb{R}$;*
3. *the geodesic joining two points m_1, m_2 is shortest among all smooth curves joining them in the manifold;*
4. *if γ, δ are two geodesics, then $f(t) = d(\gamma(t), \delta(t))$ is a convex function.*

Therefore, all these spaces are non positive curvature spaces in the sense of Alexandroff.

Example 11 Let us compute the distance $d_D(0, z)$ between 0 and z in $\mathcal{D}$. Recall from Sect. 2.7, and specifically from (32) the explicit form of the modulus of the inverse of the exponential map (based at $0 \in \mathcal{D}$):

$$|\mathbf{Log}_0(z)| = \frac{1}{2} \log\left((1+|z|)(1-|z|)^{-1}\right).$$

Then

$$d(0, z) = \|\alpha\| = \|\mathbf{Log}_0(z)\| = \frac{1}{2} \| \log\left((1+|z|)(1-|z|)^{-1}\right) \|.$$

The function $f(t) = \log(1+t) - \log(1-t)$ is strictly increasing in $[0, 1)$, with $f(0) = 0$. Thus (using that $\||z|\| = \|z\|$),

$$\| \log(1+|z|) - \log(1-|z|)\| = \max\{|f(t)| : t \in \sigma(|z|)\} = f(\|z\|).$$

Then,

$$d(0, z) = \frac{1}{2} \log\left(\frac{1+\|z\|}{1-\|z\|}\right).$$

Recall, for $z \in \mathcal{D}$, the element $g_z \in \mathcal{U}_\rho$ such that $g_z \cdot 0 = z$ given in (27),

$$g_z = \begin{pmatrix} (1-z^*z)^{-1/2} & (1-z^*z)^{-1/2}z^* \\ z(1-z^*z)^{-1/2} & (1-zz^*)^{-1/2} \end{pmatrix}.$$

Using the fact that the action of $\mathcal{B}$ on $\mathcal{D}$ is isometric, and the above construction of g_z, for arbitrary $z_1, z_2 \in \mathcal{D}$, we can compute $d(z_1, z_2)$ as follows:

$$d(z_1, z_2) = d(0, g_{z_1}^{-1} \cdot z_2) = \frac{1}{2} \log\left(\frac{1 + \|g_{z_1}^{-1} \cdot z_2\|}{1 - \|g_{z_1}^{-1} \cdot z_2\|}\right). \tag{52}$$

In order to compute g_z^{-1}, recall that elements g in $\mathcal{U}_\rho$ are characterized by the relation $\rho g^* \rho = g^{-1}$. Then

$$g_z^{-1} = \begin{pmatrix} (1-z^*z)^{-1/2} & -(1-z^*z)^{-1/2}z^* \\ -z(1-z^*z)^{-1/2} & (1-zz^*)^{-1/2} \end{pmatrix}.$$

Then, after straightforward computations,

$$g_{z_1}^{-1} \cdot z_2 = (1 - z_1 z_1^*)^{-1/2}(z_2 - z_1^*)(1 - z_1^* z_2)^{-1}(1 - z_1^* z_1)^{1/2}.$$

Remark 15 In the scalar case $\mathcal{A} = \mathbb{C}$, this norm equals

$$d(0, z) = \frac{1}{2} \log\left(\frac{1 + |z|}{1 - |z|}\right),$$

which is ($\frac{1}{2}$ times) the Poincaré distance in the open unit disk $\mathbb{D}$.

3.3 The Tangent Spaces of $\mathcal{K}_\rho$

We remarked before that the range of $q = p_{\mathbf{x}} \in \mathcal{Q}_\rho$ has basis $\{\mathbf{x}\}$ ($\mathbf{x} \in \mathcal{K}_\rho$), as right $\mathcal{A}$-submodule of $\mathcal{A}^2$. We complete this observation by showing that also $N(q)$ has a basis, and that together with $\mathbf{x}$ they form a θ_ρ-orthonormal basis of $\mathcal{A}^2$.

Proposition 23 *Let $\mathbf{x} \in \mathcal{K}_\rho$. There exist unique elements $\mathbf{y} \in R(p_{\mathbf{x}})$ and $\mathbf{z} \in N(p_{\mathbf{x}})$, $\mathbf{y}, \mathbf{z} \in \mathcal{K}_\rho$, with y_1, z_1 positive and invertible.*

Proof Straightforward computations show that

$$\mathbf{y} = \mathbf{x}x_1^*(x_1x_1^*)^{-1/2} \text{ and } \mathbf{z} = \begin{pmatrix} x_1x_1^*(1 + x_2x_2^*)^{-1/2} \\ (1 + x_2x_2^*)^{-1/2} \end{pmatrix}$$

satisfy the properties above. Let us prove that they are unique. If $\mathbf{y}' \in R(p_{\mathbf{x}}) \cap \mathcal{K}_\rho$ and y_1 is positive and invertible, then $\mathbf{y}' = \mathbf{y}a$ for some $a \in \mathcal{A}$, because $\mathbf{y}$ is a basis for $[\mathbf{x}]$. Note that $1 = \theta_\rho(\mathbf{y}', \mathbf{y}') = a^*\theta_\rho(\mathbf{y}, \mathbf{y})a = a^*a$, and that $y_1' = y_1a$. Then a is invertible, and therefore unitary. Then the identity $y_1' = y_1a$ can be regarded as

a polar decomposition of y_1', which is already positive. By uniqueness of the polar decomposition, it follows that $a = 1$. The uniqueness of $\mathbf{z}$ follows analogously. □

The unitary group $\mathcal{U}_{\mathcal{A}}$ acts on each $\mathbf{x}$ in $\mathcal{K}_\rho$. We get the tangent map of this action

$$\iota_{\mathbf{x}} : \mathcal{A}_{ah} = (T\mathcal{U}_{\mathcal{A}})_1 \to (T\mathcal{K}_\rho)_{\mathbf{x}} \tag{53}$$

which is injective and splits: one has the decomposition $(T\mathcal{K}_\rho)_{\mathbf{x}} = N(p_{\mathbf{x}}) \oplus im\ \iota_{\mathbf{x}}$.

Remark 16 $N(p_{\mathbf{x}})$ is the unique singly generated $\mathcal{A}$-submodule of $(T\mathcal{K}_\rho)_{\mathbf{x}}$.

3.3.1 A Lifting Form

This short section deals with several properties of the tangent map at $\mathbf{x} \in \mathcal{K}_\rho$ of $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$, $\varphi_\rho(\mathbf{x}) = p_{\mathbf{x}}$. Recall the tangent spaces

$$(T\mathcal{K}_\rho)_{\mathbf{x}} = \{\mathbf{z} \in \mathcal{A}^2 : \operatorname{Re} \mathbf{x}^*\rho\mathbf{z} = 0\},\ \mathbf{x} \in \mathcal{K}_\rho,$$

and

$$(T\mathcal{Q}_\rho)_q = \{a \in M_2 : aq + qa = a,\ \rho a = a^*\rho\},\ q \in \mathcal{Q}_\rho.$$

It is easy to verify that

$$(T\varphi_\rho)_{\mathbf{x}} : (T\mathcal{K}_\rho)_{\mathbf{x}} \to (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$$

is given by

$$(T\varphi_\rho)_{\mathbf{x}}\mathbf{z} = \mathbf{z}\mathbf{x}^*\rho + \mathbf{x}\mathbf{z}^*\rho,\ \mathbf{z} \in (T\mathcal{K}_\rho)_{\mathbf{x}}.$$

Notice that if $\mathbf{x} \in \mathcal{K}_\rho$ and $a \in (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$ it holds that $a\mathbf{x} \in N(p_{\mathbf{x}})$. Indeed, since $ap_{\mathbf{x}} + p_{\mathbf{x}}a = a$, it holds that $ap_{\mathbf{x}}\mathbf{x} + p_{\mathbf{x}}a\mathbf{x} = a\mathbf{x}$. Since $p_{\mathbf{x}}\mathbf{x} = \mathbf{x}$, this implies that $p_{\mathbf{x}}a\mathbf{x} = 0$.

Notice also that $N(p_{\mathbf{x}}) \subset (T\mathcal{K}_\rho)_{\mathbf{x}}$: if $\mathbf{z} \in \mathcal{A}^2$ and $\mathbf{x}\mathbf{x}^*\rho\mathbf{z} = 0$, then, since $x_1 \in G_{\mathcal{A}}$, it must be $\mathbf{x}^*\rho\mathbf{z} = 0$, a fortiori $\operatorname{Re} \mathbf{x}^*\rho\mathbf{z} = 0$.

Proposition 24 *For every $\mathbf{x} \in \mathcal{K}_\rho$ the $\mathcal{A}$-linear map*

$$\kappa_{\mathbf{x}} : (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}} \to (T\mathcal{K}_\rho)_{\mathbf{x}},\ \ \kappa_{\mathbf{x}}a = a\mathbf{x}$$

satisfies

1. $((T\varphi_\rho)_{\mathbf{x}} \circ \kappa_{\mathbf{x}})a = a$, *for all $a \in (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$;*
2. $\kappa_{\mathbf{x}} \circ (T\varphi_\rho)_{\mathbf{x}} = 1 - p_{\mathbf{x}}$;
3. $im\ \kappa_{\mathbf{x}} = N(p_{\mathbf{x}})$.

Proof 1. It is a simple computation: if $a \in (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$, then

$$((T\varphi_\rho)\kappa_{\mathbf{x}})a = (T\varphi_\rho)_{\mathbf{x}}a\mathbf{x} = a\mathbf{x}\mathbf{x}^*\rho + \mathbf{x}\mathbf{x}^*a^*\rho = ap_{\mathbf{x}} + \mathbf{x}\mathbf{x}^*\rho a,$$

because $\rho a = a^*\rho$, and then we get

$$((T\varphi_\rho)\kappa_{\mathbf{x}})a = ap_{\mathbf{x}} + p_{\mathbf{x}}a = a.$$

2. If $\mathbf{z} \in (T\mathcal{K}_\rho)_{\mathbf{x}}$, then

$$(\kappa_{\mathbf{x}} \circ (T\varphi_\rho)_{\mathbf{x}})\mathbf{z} = \kappa_{\mathbf{x}}(\mathbf{z}\mathbf{x}^*\rho + \mathbf{x}\mathbf{z}^*\rho) = (\mathbf{z}\mathbf{x}^*\rho + \mathbf{x}\mathbf{z}^*\rho)\mathbf{x} = \mathbf{z}\mathbf{x}^*\rho\mathbf{x} + \mathbf{x}\mathbf{z}^*\rho\mathbf{x} = \mathbf{z} - \mathbf{x}\mathbf{x}^*\rho\mathbf{z},$$

because $\mathbf{z}^*\rho\mathbf{x} + \mathbf{x}^*\rho\mathbf{z}^* = 0$. Then the above expression equals

$$\mathbf{z} - p_{\mathbf{x}}\mathbf{z} = (1 - p_{\mathbf{x}})\mathbf{z}.$$

3. In the previous remarks we noticed, implicitly, that $im\ \kappa_{\mathbf{x}} \subset N(p_{\mathbf{x}})$. On the other hand, going backwards in the proof of 2., we get, for $\mathbf{y} \in N(p_{\mathbf{x}})$, that

$$\mathbf{y} = (1 - p_{\mathbf{x}})\mathbf{z} = \mathbf{z} - \mathbf{x}\mathbf{x}^*\rho\mathbf{z} = \mathbf{z} + \mathbf{x}\mathbf{z}^*\rho\mathbf{x} = \mathbf{z}\mathbf{x}^*\rho\mathbf{x} + \mathbf{x}\mathbf{z}^*\rho\mathbf{x} = (\mathbf{z}\mathbf{x}^*\rho + \mathbf{x}\mathbf{z}^*\rho)\mathbf{x} \in im\ \kappa_{\mathbf{x}}.$$

□

Definition 9 We call $\kappa_{\mathbf{x}}$ the lifting form of $\mathcal{Q}_\rho$.

Corollary 9 *For $\mathbf{x} \in \mathcal{K}_\rho$, there is a decomposition*

$$(T\mathcal{K}_\rho)_{\mathbf{x}} = N(p_{\mathbf{x}}) \oplus N((T\varphi_\rho)_{\mathbf{x}}).$$

For further reference, we write

$$\mathbf{H}_{\mathbf{x}} = N(p_{\mathbf{x}}), \tag{54}$$

and call it the *horizontal space* over $\mathbf{x}$.

Corollary 10 *For $\mathbf{x} \in \mathcal{K}_\rho$, the restriction*

$$(T\varphi_\rho)_{\mathbf{x}}\big|_{N(p_{\mathbf{x}})} : N(p_{\mathbf{x}}) \to (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$$

is a linear isomorphism with inverse $\kappa_{\mathbf{x}}$.

Remark 17 If $a \in (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$ and $\mathbf{y} \in \mathcal{K}_\rho$ is such that $p_{\mathbf{y}} = p_{\mathbf{x}}$, then $\mathbf{y} = \mathbf{x}u$ for some $u \in \mathcal{U}_{\mathcal{A}}$ (see Lemma 2). Then $\kappa_{\mathbf{y}}(a) = a\mathbf{y} = a\mathbf{x}u$. Therefore, we can think of every $a \in (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$ as the set of pairs $(\mathbf{x}u, \mathbf{z}u)$ with $\mathbf{z} \in N(p_{\mathbf{x}})$ and $u \in \mathcal{U}_{\mathcal{A}}$. We write $(\mathbf{x}, \mathbf{z}) \sim (\mathbf{x}u, \mathbf{z}u)$. In Sect. 1.1.4 of Chap. 4 we will show that this set of pairs is precisely the $\mathcal{K}_\rho$-presentation of the tangent bundle $T\mathcal{Q}_\rho$, in the sense of Definition 1 in Chap. 4 below.

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The Poincaré Half-Space of a C^*-Algebra

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Abstract In this chapter we continue our study of the space $\mathcal{H}$ and its alternative presentations as $\mathcal{Q}_\rho$ and $\mathcal{D}$. Regarded as $\mathcal{D}$, the projective nature of $\mathcal{H}$ arises: it is using this model that we can characterize the limit points of geodesics, and prove one of our main results: the (operator valued) cross-ratio of the four-tuple $\delta(-\infty), \delta(0) = z_0, \delta(1) = z_1, \delta(+\infty)$ induced by a geodesic δ of $\mathcal{D}$, coincides with the logarithm based at z_0 evaluated at the point z_1 (as happens in the classic scalar Poincaré disk). We introduce in $\mathcal{H}$ an operator valued Kähler structure. This, in turn, induces on $\mathcal{H}$ a homogeneous symplectic structure, which is the imaginary part of the Hilbertian form. We show that this symplectic structure is the covariant differential of the Liouville 1-form of $\mathcal{H}$. We compute the moment map. In the presence of a trace in $\mathcal{A}$, we show that, for an appropriate subgroup of $\mathcal{U}(\theta)$, the image of the moment map is a convex set, which resembles the classical result of Atiyah, Guillemin and Sternberg.

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1 Vector Bundles

1.1 The Coefficient Bundle

Let us describe a setting, which will allow us to present a useful approach to the three main vector bundles which we shall use to study the geometry of $\mathcal{Q}_\rho$. They are all based on the principal bundle $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$.

Let $\eta : F \to \mathcal{Q}_\rho$ be a fiber bundle, consider the commutative diagram

$$\begin{array}{ccc} F_{\mathcal{K}_\rho} & \xrightarrow{\tilde{\varphi_\rho}} & F \\ \eta^* \downarrow & & \downarrow \eta \,, \\ \mathcal{K}_\rho & \xrightarrow{\varphi_\rho} & \mathcal{Q}_\rho \end{array} \tag{1}$$

where $F_{\mathcal{K}_\rho} = \{(f, \mathbf{x}) : f \in F,\ \mathbf{x} \in \mathcal{K}_\rho,\ \eta(f) = \varphi_\rho(\mathbf{x})\}$. This diagram describes η^* as the bundle over $\mathcal{K}_\rho$ induced by φ_ρ. Note that $F_{\mathcal{K}_\rho}$ carries a natural action from $\mathcal{U}_\mathcal{A}$, which lifts the action of $\mathcal{U}_\mathcal{A}$ over $\mathcal{K}_\rho$, and which presents the bundle $\eta : F \to \mathcal{Q}_\rho$ as the space of orbits of $\eta^* : F_{\mathcal{K}_\rho} \to \mathcal{K}_\rho$ under this action.

We can argue that an element $f \in F$ over $q \in \mathcal{Q}_\rho$ is represented by an element $\tilde{f} \in F_{\mathcal{K}_\rho}$ in the basis $\mathbf{x} \in \mathcal{K}_\rho$ ($\varphi_\rho(\mathbf{x}) = q$, f lies over the same $\mathbf{x}$). More precisely, $(\tilde{f}, \mathbf{x})$ and $(\tilde{f}', \mathbf{x}')$ represent the same element if and only if there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\tilde{f}' = \tilde{f}u$ and $\mathbf{x}' = \mathbf{x}u$.

Definition 1 Given a bundle $\eta : F \to \mathcal{Q}_\rho$, we call a $\mathcal{K}_\rho$-*presentation* of η a bundle $\zeta : Z \to \mathcal{K}_\rho$, with an action of $\mathcal{U}_\mathcal{A}$ over F which lifts the action of $\mathcal{U}_\mathcal{A}$ over $\mathcal{K}_\rho$, and a bundle isomorphism $\Phi : Z \to F_{\mathcal{K}_\rho}$ preserving the action of $\mathcal{U}_\mathcal{A}$,

$$\begin{array}{ccc} Z & \xrightarrow{\Phi} & F_{\mathcal{K}_\rho} \\ \zeta \searrow & & \swarrow \eta \,. \\ & \mathcal{K}_\rho & \end{array}$$

Let us give three examples of $\mathcal{K}_\rho$-presentations which will be relevant in this paper.

1.1.1 The *Tautological Bundle* Over $\mathcal{Q}_\rho$

Consider

$$\xi : \mathcal{E} \to \mathcal{Q}_\rho,$$

where $\mathcal{E} = \{(q, \mathbf{x}) \in M_2(\mathcal{A}) \times \mathcal{A}^2 : q \in \mathcal{Q}_\rho, \mathbf{x} \in R(q)\}$ and $\xi(q, \mathbf{x}) = q$. Over each idempotent q we put the vectors in the range of q.

Proposition 1 *ξ is a locally trivial vector bundle.*

Proof Fix $q_0 \in \mathcal{Q}_\rho$. The map $\pi_{q_0} : \mathcal{U}_\rho \to \mathcal{Q}_\rho$, $\pi_{q_0}(\tilde{a}) = \tilde{a} q_0 \tilde{a}^{-1}$ has smooth local cross sections: there exists $r_{q_0} > 0$ and a map

$$g_{q_0} : \mathcal{W}_{q_0} = \{q \in \mathcal{Q}_\rho : \|q - q_0\| < r_{q_0}\} \to \mathcal{U}_\rho,$$

such that $g(q) q_0 g^{-1}(q) = q$ (see [1]). Consider the set $\mathcal{V}_{q_0} = \{(q, \mathbf{x}) \in \xi : q \in \mathcal{W}_{q_0}\}$. Clearly $\mathcal{V}_{q_0}$ is open in ξ. The map

$$\Phi_{q_0} : \mathcal{V}_{q_0} \to \mathcal{W}_{q_0} \times R(q_0)\ , \quad \Phi_{q_0}(q, \mathbf{x}) = (q, g^{-1}(q)\mathbf{x})$$

is a local trivialization for ξ near q_0. □

Then we have the induced bundle over $\mathcal{K}_\rho$, and the diagram (1) in this case:

$$\begin{array}{ccc} \mathcal{E}_{\mathcal{K}_\rho} & \xrightarrow{\tilde{\varphi_\rho}} & \mathcal{E} \\ \xi^* \downarrow & & \downarrow \xi\,, \\ \mathcal{K}_\rho & \xrightarrow{\varphi_\rho} & \mathcal{Q}_\rho \end{array}$$

Consider the product bundle $\mathcal{K}_\rho \times \mathcal{A} \to \mathcal{A}$ and the isomorphism $\beta : \mathcal{K}_\rho \times \mathcal{A} \to \mathcal{E}$, $\beta(\mathbf{x}, a) = (p_{\mathbf{x}}, \mathbf{x}a)$. The action of $\mathcal{U}_\mathcal{A}$ on $\mathcal{K}_\rho \times \mathcal{A}$ is given by $(\mathbf{x}, a) \cdot u = (\mathbf{x}u, u^* a)$. The isomorphism β gives a $\mathcal{K}_\rho$-presentation for $\mathcal{E}$. Thus, an element $(q, \mathbf{y})$ of $\mathcal{E}$ is represented by a pair $(\mathbf{x}, a)$, and two pairs $(\mathbf{x}, a)$, $(\mathbf{x}', a')$ represent the same element of $\mathcal{E}$ if and only if there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}' = \mathbf{x}u$ and $a' = u^* a$.

1.1.2 The *Coefficient* Bundle

The fiber bundle defined now will be a central feature of our exposition. It will provide a setting for the most relevant notions studied in this paper.

We shall describe a bundle over $\mathcal{Q}_\rho$, whose fiber over q is the space of right $\mathcal{A}$-module endomorphisms of $R(q)$, which we shall call the *coefficient bundle*

$$\gamma : C \to \mathcal{Q}_\rho.$$

Formally, $C = \{(q, \varphi) \in \mathcal{Q}_\rho \times \mathcal{L}_\mathcal{A}(R(q)) : q \in \mathcal{Q}_\rho\}$, where $\mathcal{L}_\mathcal{A}(R(q))$ denotes the space of right $\mathcal{A}$-module endomorphisms of the module $R(q)$, and the map $\gamma : C \to \mathcal{Q}_\rho$ is $\gamma(q, \varphi) = q$. Given $\mathbf{x} \in \mathcal{K}_\rho$ with $p_\mathbf{x} = q$, an endomorphism φ of $R(q)$ gives $\varphi(\mathbf{x}) = \mathbf{x}a$. We say that $\mathbf{x}$ is a *basis* for $R(q)$, and that $a \in \mathcal{A}$ is the *matrix* of φ in the basis $\mathbf{x}$. If we change $\mathbf{x}$ for $\mathbf{y} = \mathbf{x}u \in R(q)$ for $u \in \mathcal{U}_\mathcal{A}$, and denote by b the matrix of φ at $\mathbf{y}$, $\varphi(\mathbf{y}) = \mathbf{y}b$, then

$$\mathbf{x}ub = \mathbf{y}b = \varphi(\mathbf{y}) = \varphi(\mathbf{x})u = \mathbf{x}au, \tag{2}$$

i.e., $b = u^*au$. In other words, the endomorphisms can be identified with the pairs $(\mathbf{x}, a) \in \mathcal{K}_\rho \times \mathcal{A}$, subject to the equivalence relation

$$(\mathbf{x}, a) \sim (\mathbf{x}u, u^*au)\ , \quad u \in \mathcal{U}_\mathcal{A}. \tag{3}$$

Notation 1 We denote by $\Big[(\mathbf{x}, a)\Big]$ the endomorphism of the module $R(p_\mathbf{x})$ which has matrix a in the basis $\mathbf{x}$.

Remark 1 Observe that C is a fiber bundle whose fibers are C*-algebras. If, additionally, $\mathcal{A}$ is commutative then C is the product bundle $\mathcal{Q}_\rho \times \mathcal{A}$.

Proposition 2 *The space C is a C^∞ differentiable manifold and the map $\gamma : C \to \mathcal{Q}_\rho$ is a C^∞ trivial fiber bundle.*

Proof We define a smooth structure on C by constructing local charts. Fix $(q_0, \varphi_0) \in C$. Consider the set

$$\mathcal{W}_{q_0} = \{(q, \varphi) \in C : \|q - q_0\| < r_{q_0}\},$$

where r_{q_0} is the radius given in the proof of Proposition 1. There exists a map g_{q_0} defined on such idempotents q, with values in $\mathcal{U}_\rho$, such that $g_{q_0}(q)q_0g_{q_0}^{-1}(q) = q$. Note that this implies that the $\mathcal{A}$-module homomorphism g_{q_0} (acting in $\mathcal{A}^2$) maps $R(q_0)$ onto $R(q)$. Then $g_{q_0}\varphi g_{q_0}^{-1}$ is a module homomorphism acting in $R(q_0)$. The map $\mathcal{W}_{q_0} \to \mathcal{W}_0$, where

$$\mathcal{W}_0 = \{(q, \psi) : q \in \mathcal{Q}_\rho, \|q - q_0\| < r_{q_0}, \psi \in \mathcal{L}_\mathcal{A}(R(q_0))\}\ , \ (q, \varphi) \mapsto (q, g_{q_0}\varphi g_{q_0}^{-1}),$$

is one to one. The set $\mathcal{W}_0$ is clearly a C^∞ manifold, because it is an open subset of $\mathcal{Q}_\rho \times \mathcal{L}_\mathcal{A}(R(q_0))$, which is itself a product of a manifold and a Banach space. If $\mathcal{W}_{q_0} \cap \mathcal{W}_{q_1} \neq \emptyset$, the transition map between $\mathcal{W}_0$ and $\mathcal{W}_1$, which is given by

$$\mathcal{W}_0 \ni (q, \psi) \mapsto (q, g_{q_1}g_{q_0}^{-1}\psi g_{q_0}g_{q_1}^{-1}) \in \mathcal{W}_1,$$

is clearly C^∞. This defines a C^∞ structure on C, which makes the projection map $\gamma : C \to \mathcal{Q}_\rho$ a C^∞ map.

Let us prove that it is a trivial bundle. The map $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$ has a global cross section $\sigma_\rho : \mathcal{Q}_\rho \to \mathcal{K}_\rho$ defined by $\sigma_\rho(q) = \lambda_q \mathbf{e}_1$ (see Theorem 4, part 1 in Chap. 3).

Consider the map $\Phi : C \to \mathcal{Q}_\rho \times \mathcal{A}, \Phi(q, \varphi) = (q, a)$, where $a \in \mathcal{A}$ is the matrix of φ in the basis $\mathbf{w} = \varphi_\rho(q)$ of q. This gives a global trivialization for γ. □

As in the previous cases, the coefficient bundle $\gamma : C \to \mathcal{Q}_\rho$ induces the bundle $\gamma^* : C_{\mathcal{K}_\rho} \to \mathcal{K}_\rho$ and the commutative diagram (1) in this setting becomes

$$\begin{array}{ccc} C_{\mathcal{K}_\rho} & \xrightarrow{\tilde{\varphi_\rho}} & C \\ \gamma^* \downarrow & & \downarrow \gamma\ . \\ \mathcal{K}_\rho & \xrightarrow{\varphi_\rho} & \mathcal{Q}_\rho \end{array}$$

The fiber $(C_{\mathcal{K}_\rho})_{\mathbf{x}}$ over $\mathbf{x} \in \mathcal{K}_\rho$ consists of the endomorphisms of $R(p_{\mathbf{x}})$. Therefore, one has the representation $\Phi : \mathcal{K}_\rho \times \mathcal{A} \to C_{\mathcal{K}_\rho}$, given by

$$\Phi(\mathbf{x}, a) = (\mathbf{x}, [(\mathbf{x}, a)]).$$

1.1.3 The Connection in the Coefficient Bundle

Since C_q consists of endomorphisms ξ_q, $q \in \mathcal{Q}_\rho$, the standard connection on C is given by the *Leibnitz formula*. Let $\mathbf{X} \in T\mathcal{Q}_\rho$ and $\mathbf{x} = \mathbf{x}(t)$ adjusted to $\mathbf{X}$. Let σ and φ be cross sections of ξ and C, respectively. We define the covariant derivative $D_{\mathbf{X}}\varphi$ by the rule

$$(D_{\mathbf{X}}\varphi)\sigma := D_{\mathbf{X}}(\varphi\sigma) - \varphi(D_{\mathbf{X}}\sigma). \tag{4}$$

Alternatively, write $\sigma(t) = \mathbf{x}(t)a(t)$ and $\varphi_t(\mathbf{z}) = \varphi_t(\mathbf{x}b) = \mathbf{x}(t)\lambda_t\, b$ for $\lambda, b \in \mathcal{A}$. Let us compute both terms in (4).

$$D_{\mathbf{x}}(\varphi\sigma) = D_{\mathbf{x}}(\varphi(\mathbf{x}a)) = D_{\mathbf{x}}(\mathbf{x}\lambda a) = \mathbf{x}\left(\dot{\lambda}a + \lambda\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})\lambda a\right)$$

On the other hand

$$\varphi(D_{\mathbf{x}}\sigma) = \varphi(\mathbf{x}\dot{a} + p_{\mathbf{x}}(\dot{\mathbf{x}})a) = \mathbf{x}\lambda\dot{a} + \varphi(p_{\mathbf{x}}(\dot{\mathbf{x}})a.$$

Put $p_{\mathbf{x}}(\dot{\mathbf{x}}) = \mathbf{x}b$, where $b = \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})$, and then

$$\varphi(D_{\mathbf{x}}\sigma) = \mathbf{x}\lambda\dot{a} + \mathbf{x}\lambda\theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a.$$

Subtracting as in (4), one gets

$$(D_{\mathbf{x}}\varphi)\sigma = \mathbf{x}\left(\dot{\lambda} + [\theta_\rho(\mathbf{x}, \dot{\mathbf{x}}), \lambda]\right)a = \mathbf{x}\left(\dot{\lambda} + [\mathbf{x}^*\rho\mathbf{x}, \lambda]\right)a. \tag{5}$$

1.1.4 The Tangent Bundle $T\mathcal{Q}_\rho$

For any $\mathbf{x} \in \mathcal{K}_\rho$, the nullspace $N(p_{\mathbf{x}})$ is a subspace of $(T\mathcal{K}_\rho)_{\mathbf{x}}$, which we called the *horizontal space* $\mathbf{H}_{\mathbf{x}}$. The tangent map of the fibration $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$

$$(T\varphi_\rho)_{\mathbf{x}} : (T\mathcal{K}_\rho)_{\mathbf{x}} \to (T\mathcal{Q}_\rho)_{p_{\mathbf{x}}}$$

is bijective when restricted to $\mathbf{H}_{\mathbf{x}}$. Then we have the bundle $\nu : \mathbf{H} \to \mathcal{K}_\rho$, where $\mathbf{H} = \{(\mathbf{z}, \mathbf{x}) : \mathbf{x} \in \mathcal{K}_\rho,\ \mathbf{z} \in \mathbf{H}_{\mathbf{x}}\}$, and the map is given by $\nu(\mathbf{z}, \mathbf{x}) = \mathbf{x}$. Thus, we have the $\mathcal{K}_\rho$-presentation

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{\Phi} & (TQ_\rho)_{\mathcal{K}_\rho} \\ {\scriptstyle \nu}\searrow & & \swarrow{\scriptstyle \tau^*} \\ & \mathcal{K}_\rho & \end{array},$$

where $\tau : TQ_\rho \to Q_\rho$ is the tangent bundle and $\tau^* : (TQ_\rho)_{\mathcal{K}_\rho} \to \mathcal{K}_\rho$ is the induced bundle. The isomorphism Φ is given by the differential of φ_ρ restricted to $\mathbf{H}$, namely

$$\Phi(\mathbf{x}, \mathbf{z}) = (\mathbf{x}, (d\varphi_\rho)_{\mathbf{x}}(\mathbf{z})) \ , \ \text{for } \mathbf{x} \in \mathcal{K}_\rho \ , \ \mathbf{z} \in \mathbf{H}_{\mathbf{x}}. \tag{6}$$

Then a tangent vector to Q_ρ at $p_{\mathbf{x}}$ is represented by a pair $(\mathbf{x}, \mathbf{z})$, $\mathbf{x} \in \mathcal{K}_\rho$, $\mathbf{z} \in \mathbf{H}_{\mathbf{x}}$; $(\mathbf{x}, \mathbf{z})$ and $(\mathbf{x}u, \mathbf{z}')$ represent the same vector if and only if $\mathbf{z}' = \mathbf{z}u$ ($u \in \mathcal{U}_{\mathcal{A}}$).

Remark 2 We may consider the *tautological* 1-form with values in the tangent bundle TQ_ρ that associates to a tangent vector $\mathbf{X}$ to Q_ρ, the same vector $\mathbf{X}$ as an element of TQ_ρ. The lifting form given in Definition 9 of Chap. 3 is just an expression of the values of this tautological form in terms of the principal bundle $\mathcal{K}_\rho$.

1.1.5 TQ_ρ as a C-Module

Recall from (4) that the fiber C_q can be presented as the set of pairs $(\mathbf{x}, a) \in \mathcal{K}_\rho \times \mathcal{A}$ modulo the equivalence $(\mathbf{x}, a) \sim (\mathbf{x}u, u^*au)$ for $u \in \mathcal{U}_{\mathcal{A}}$. Since each submodule $R(q)$ has a basis consisting of one element, then $\mathcal{L}_{\mathcal{A}}(R(q))$ is in one to one correspondence with $\mathcal{A}$: having chosen a basis $\mathbf{x} \in \mathcal{K}_\rho$ for $R(q)$, the map $\mathcal{A} \to C_q$, $a \mapsto (\mathbf{x}, a)$ is a $*$-isomorphism. Thus, each fiber C_q has the structure of a C*-algebra, namely $\mathcal{A}$. Clearly the structure does not depend on the choice of the basis; another basis $\mathbf{x}u$ provides the same structure, because $Ad(u)$ is a $*$-automorphism of $\mathcal{A}$.

On the other hand, we saw before that the tangent bundle has a similar presentation. Each $\mathbf{X} \in (TQ_\rho)_q$ is given by the set of pairs $(\mathbf{x}, \mathbf{v}) \in \mathcal{K}_\rho \times \mathcal{A}^2$, modulo the equivalence $(\mathbf{x}, \mathbf{v}) \sim (\mathbf{x}u, \mathbf{v}u)$ for $u \in \mathcal{U}_{\mathcal{A}}$ and $\mathbf{v} = \mathbf{X}\mathbf{x} = \kappa_{\mathbf{x}}(\mathbf{X})$. Accordingly, we denote $\mathbf{X} \in (TQ_\rho)_q$ as $[(\mathbf{x}, \mathbf{v})]$.

Definition 2 For $\mathbf{X} \in (TQ_\rho)_q$ and $\varphi \in \mathcal{L}_{\mathcal{A}}(R(q))$ we define the product

$$\mathbf{X} \cdot \varphi := [(\mathbf{x}, \mathbf{v}a)] \in (TQ_\rho)_q$$

if $\mathbf{x} \in \mathcal{K}_\rho$, $p_{\mathbf{x}} = q$, $\varphi = [(\mathbf{x}, a)]$ and $\mathbf{X} = [(\mathbf{x}, \mathbf{v})]$.

Observe that this operation is well defined: if we change the basis to $\mathbf{x}u$, then φ and $\mathbf{X}$ are represented by $(\mathbf{x}u, u^*au)$ and $(\mathbf{x}u, \mathbf{v}u)$ respectively, and the product in the new referential is $(\mathbf{x}u, \mathbf{v}uu^*au) = (\mathbf{x}u, \mathbf{v}au)$ which represents the same tangent vector $\mathbf{X} \cdot \varphi$ as $(\mathbf{x}, \mathbf{v}a)$.

Thus, TQ_ρ can be regarded as a bundle of right modules over C, which is a bundle of C*-algebras (isomorphic to $\mathcal{A}$).

Remark 3 *The tautological bundle over $\mathcal{K}_\rho$.*
Let us consider the pullback $\mathcal{E}' = \varphi_\rho^* \mathcal{E}$ of $\mathcal{E}$ by φ_ρ,

$$\xi' : \mathcal{E}' = \{(\mathbf{x}, \mathbf{v}) \in \mathcal{K}_\rho \times \mathcal{A}^2 : p_{\mathbf{x}}\mathbf{v} = \mathbf{v}\} \to \mathcal{K}_\rho \ , \ \ \xi'(\mathbf{x}, \mathbf{v}) = \mathbf{x}.$$

Pick $\mathbf{x} \in \mathcal{K}_\rho$ and let $\mathcal{E}'_{\mathbf{x}}$ be the fiber of $\mathcal{X}'$ over $\mathbf{x}$. Then the map

$$\mathcal{A} \to \mathcal{E}'_{\mathbf{x}} \ , \ \ a \mapsto \mathbf{x}a$$

is an isomorphism of right $\mathcal{A}$-modules, with inverse $\mathbf{v} \mapsto \theta_\rho(\mathbf{x}, \mathbf{v})$. It can be regarded as a chart for $\mathcal{E}'_{\mathbf{x}}$ associated to $\mathbf{x}$. Let $\mathbf{y} \in \mathcal{E}_{p_{\mathbf{x}}}$ (i.e., $p_{\mathbf{x}} = p_{\mathbf{y}} = q$); then there exists a unitary $u \in \mathcal{U}_{\mathcal{A}}$ such that $\mathbf{y} = \mathbf{x}u$. The isomorphism between $\mathcal{A}$ and $\mathcal{E}'_{\mathbf{y}}$ is $\mathcal{A} \ni b \mapsto \mathbf{y}a$. The coordinates a and b of a given $\mathbf{v} \in \mathcal{E}_q$ are related by $\mathbf{x}ub = \mathbf{y}b = \mathbf{v} = \mathbf{x}a$, and thus, $a = ub$. Then we can regard the vectors $\mathbf{v} \in \mathcal{E}_q$ as pairs $(\mathbf{x}, a) \in \mathcal{K}_\rho \times \mathcal{A}$, with the following equivalence relation:

$$(\mathbf{x}, a) \sim (\mathbf{x}u, u^* a) \ , \ \ u \in \mathcal{U}_{\mathcal{A}}.$$

Remark 4 *The connection in the tautological bundle.*
The covariant derivative in the tautological bundle $\xi : \mathcal{E} \to \mathcal{Q}_\rho$ is given by

$$D_{\mathbf{X}}\sigma = q(\mathbf{X} \bullet \sigma),$$

where σ is a cross section for ξ and $\mathbf{X} \bullet \sigma$ is the directional derivative of σ, considering σ as a function with values in $\mathcal{A}^2$. Let us write this explicitly, in referential terms: let $q(t)$ be a smooth curve in $\mathcal{Q}_\rho$, with $\dot{q} = \mathbf{X}$, and let $\mathbf{x}(t)$ be a smooth lifting of $q(t)$ in $\mathcal{K}_\rho$, i.e, $p_{\mathbf{x}(t)} = q(t)$. Then $\sigma(t) = \mathbf{x}(t)a(t)$ where $a(t) \in \mathcal{A}$ is a smooth curve. We have

$$D_{\mathbf{x}}\sigma = \frac{d}{dt} q(\mathbf{x}(t)a(t)) = q(\dot{\mathbf{x}}(t)a(t) + \mathbf{x}(t)\dot{a}(t)) = \mathbf{x}\left(\dot{a}(t) + \theta_\rho(\mathbf{x}(t), \dot{\mathbf{x}}(t))a(t)\right).$$

This formula can be read as follows: the covariant derivative has two terms in the fibers of ξ_q, $\mathbf{x}a$ and the component of $\dot{\mathbf{x}}$ in $R(q)$ multiplied by a. Note also that since $q\mathbf{x} = \mathbf{x}$, $\dot{\mathbf{x}}$ decomposes as $q\mathbf{x}$ in $R(q)$ plus $\mathbf{X}\mathbf{x}$ in $N(q)$.

2 Kähler Structure

2.1 The Complex Structure of $\mathcal{Q}_\rho$

Notice that, for any $q \in \mathcal{Q}_\rho$, the tangent space $(T\mathcal{Q}_\rho)_q$ is a *real* vector space. In this section, we shall define a generalized complex structure on $\mathcal{Q}_\rho$. This means a smooth map $q \mapsto \mathcal{J}_q$, where

$$\mathcal{J}_q : (T\mathcal{Q}_\rho)_q \to (T\mathcal{Q}_\rho)_q \tag{7}$$

is a bounded linear map such that $\mathcal{J}_q^2 = -1_{(T\mathcal{Q}_\rho)_q}$, with an integrability property. For general ideas about complex structures in finite dimensional manifolds, we refer the reader to [2].

Recall, from Theorem 4 in Chap. 3, the map

$$\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho,\ \varphi_\rho(\mathbf{x}) = \mathbf{x}\mathbf{x}^*\rho$$

and its tangent map $(T\varphi_\rho)_{\mathbf{x}}\mathbf{Y} = (\mathbf{Y}\mathbf{x}^* + \mathbf{x}\mathbf{Y}^*)\rho$.

Fix $q \in \mathcal{Q}_\rho$ and $\mathbf{x} \in \mathcal{K}_\rho$ such that $q = p_{\mathbf{x}}$. Then, for all $\mathbf{Z} \in (T\mathcal{Q}_\rho)_q$, it holds that

$$(T\varphi_\rho)_{\mathbf{x}}\, i\mathbf{Z}\mathbf{x} = i\ (\mathbf{Z}\mathbf{x}\mathbf{x}^* - \mathbf{x}\mathbf{x}^*\mathbf{Z}^*)\rho = i\ (\mathbf{Z}q - q\mathbf{Z}) = i\ \mathbf{Z}(2q-1),$$

because $q = p_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*\rho$, $\mathbf{Z}^*\rho = \rho\mathbf{Z}$ and $\mathbf{Z}q + q\mathbf{Z} = \mathbf{Z}$. This shows that

$$\mathcal{J}_q\mathbf{Z} := (T\varphi_\rho)_{\mathbf{x}}\, i\ \mathbf{Z}_{\mathbf{x}} = i\ \mathbf{Z}(2q-1)$$

is a well defined map $\mathcal{J}_q : (T\mathcal{Q}_\rho)_q \to (T\mathcal{Q}_\rho)_q$. The proof that $\mathcal{J}_q(\mathcal{J}_q\mathbf{Z}) = -\mathbf{Z}$ is a simple computation.

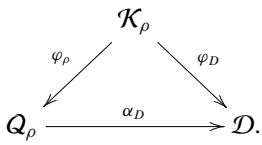
Remark 5 In the particular case when $q = p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\mathbf{x} = \mathbf{e}_1$, we have, in matrix form, $\mathbf{Z} = \begin{pmatrix} 0 & Z_1 \\ -Z_1^* & 0 \end{pmatrix}$, and

$$\mathcal{J}_p\mathbf{Z} = i\mathbf{Z}\rho = \begin{pmatrix} 0 & -iZ_1 \\ -iZ_1^* & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & Z_1 \\ Z_1^* & 0 \end{pmatrix}.$$

In order to see that this complex structure is integrable we proceed as follows. First, we note that it is invariant under the action of the group $\mathcal{U}_\rho$. Next, we identify $\mathcal{Q}_\rho$ with the disk $\mathcal{D} = \{a \in \mathcal{A} : \|a\| < 1\} \subset \mathcal{A}$. The disk $\mathcal{D}$ has a natural complex structure, as an open subset of the Banach space $\mathcal{A}$, which is invariant for the action of $\mathcal{U}_\rho$ (the identification $\mathcal{Q}_\rho \simeq \mathcal{D}$ is $\mathcal{U}_\rho$-equivariant). In order to prove that our complex structure is *integrable*, it suffices to show that these structures coincide at the point $q = p$, which corresponds to $0 \in \mathcal{D}$ in the identification $\mathcal{Q}_\rho \simeq \mathcal{D}$:

Lemma 1 *The complex multiplication defined in $\mathcal{Q}_\rho$ corresponds, under the identification $\mathcal{Q}_\rho \simeq \mathcal{D}$, to the usual multiplication by the imaginary constant i.*

Proof As noted in the remark above, it suffices to prove this fact at $0 \in \mathcal{D}$, which corresponds to $p \in \mathcal{Q}_\rho$. Recall the coordinate diagram (28) in Chap. 3:



Both $0 \in \mathcal{D}$ and $p \in \mathcal{Q}_\rho$ lift to $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{K}_\rho$. Let $\zeta \in (T\mathcal{D})_0 = \mathcal{A}$, and $\mathbf{Z} \in (T\mathcal{Q}_\rho)_p$ corresponding to the same horizontal tangent vector $\mathbf{v} = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}$ in $N(p) \subset (T\mathcal{K}_\rho)_{\mathbf{e}_1}$. By the definition given above (7), multiplication by i in $(T\mathcal{Q}_\rho)_p$ consists in the usual multiplication $i\mathbf{v} = \begin{pmatrix} 0 \\ i\,\zeta \end{pmatrix} \in N(p)$. Let us verify that this induces the usual multiplication by i in $\mathcal{A}$. The tangent map $(T\varphi_D)_{\mathbf{e}_1}$ of $\varphi_D : \mathcal{K}_\rho \to \mathcal{D}, \varphi_D(\mathbf{x}) = x_2 x_1^{-1}$ at $\mathbf{e}_1$, for $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, is given by

$$(T\varphi_D)_{\mathbf{e}_1}(\mathbf{w}) = w_2.$$

Therefore $i\mathbf{v}$ is mapped to $i\zeta$, the usual multiplication by i in $\mathcal{A}$. □

In [3, 4] the reader will find a similar treatment of complex structures in a C*-algebra context.

2.2 The Hilbertian Product in $\mathcal{Q}_\rho$

We define now on $\mathcal{Q}_\rho$ a generalization of the notion of Hermitian structure. It consists of a Hilbertian product on each $(T\mathcal{Q}_\rho)_q$ with values in C_q, which is invariant under the action of $\mathcal{U}_\rho$.

Definition 3 Given $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$, we put

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = \text{ the endomorphism of } R(q) \text{ given by the pair } (\mathbf{x}, -\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{x}}(\mathbf{Y})))$$

$$= \Big[(\mathbf{x}, -\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{x}}(\mathbf{Y}))) \Big],$$

where $\mathbf{x} \in \mathcal{K}_\rho$ is such that $p_{\mathbf{x}} = q$, and the brackets denote the equivalence class as in (3).

Note that (regarding $\mathbf{X}$ and $\mathbf{Y}$ as matrices in $M_2(\mathcal{A})$)

$$-\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{x}}(\mathbf{Y})) = -(\mathbf{X}\mathbf{x})^* \rho(\mathbf{Y}\mathbf{x}) = -\mathbf{x}^* \mathbf{X}^* \rho \mathbf{Y} \mathbf{x}.$$

Summarizing,

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = \Big[(\mathbf{x}, -\mathbf{x}^* \mathbf{X}^* \rho \mathbf{X} \mathbf{x}) \Big]. \tag{8}$$

Remark 6 Some remarks are in order.

1. The minus sign is needed so that the form is positive. Indeed, recall that θ_ρ is negative in $N(q)$ and $\mathbf{X}\mathbf{x} \in N(q)$,

$$\langle \mathbf{X}, \mathbf{X} \rangle_q = [(\mathbf{x}, -\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{X}\mathbf{x}))].$$

   Recall that the C*-algebra structure of the fibers C_q of C is that of $\mathcal{A}$: an endomorphism φ of $\mathcal{L}_\mathcal{A}(R(q) \simeq \mathcal{A}$ is positive if and only if any of its matrices is a positive element of $\mathcal{A}$. Recall also from the definition of $\mathcal{Q}_\rho$, that the projection q decomposes θ_ρ, i.e., θ_ρ is positive in $R(q)$ and negative in $N(q)$. On the other hand, as seen above, $\mathbf{X}\mathbf{x} \in N(q)$.
2. If one changes $\mathbf{x}$ for $\mathbf{x}u$ for some $u \in \mathcal{U}_\mathcal{A}$, then

$$\theta_\rho(\kappa_{\mathbf{x}u}(\mathbf{X}u), \kappa_{\mathbf{x}u}(\mathbf{Y}u)) = u^* \theta_\rho(\kappa_\mathbf{x}(\mathbf{X}), \kappa_\mathbf{x}(\mathbf{Y}))u,$$

   i.e. $\langle \mathbf{X}, \mathbf{Y} \rangle_q$ is indeed an element of C_q.
3. This Hilbertian product is linear with respect to the product defined in Definition 2: if $q = p_\mathbf{x} \in \mathcal{Q}_\rho$, $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$ and $\varphi \in \mathcal{L}_\mathcal{A}(R(q))$, then

$$\langle \mathbf{X}, \mathbf{Y} \cdot \varphi \rangle_q = \langle \mathbf{X}, \mathbf{Y} \rangle_q \cdot \varphi.$$

   Indeed, if we use the basis $\mathbf{x} \in \mathcal{K}_\rho$ for $R(q)$, then $\mathbf{Y} \cdot \varphi$ is represented by the pair $(\mathbf{x}, \mathbf{Y}\mathbf{x}a)$, and

$$\theta_\rho(\kappa_\mathbf{x}(\mathbf{X}), \kappa_\mathbf{x}(\mathbf{Y} \cdot \varphi)) = (\mathbf{X}\mathbf{x})^* \rho \mathbf{Y}\mathbf{x}a,$$

   which is the element of $\mathcal{A}$ that one uses to define $\langle \mathbf{X}, \mathbf{Y} \rangle_q \cdot \varphi$ (in terms of the basis $\mathbf{x}$).
4. Recall from the end of Sect. 3.3.1 in Chap. 3, that if $\dot{v}, \dot{w} \in (T\mathcal{H})_\zeta$ for some $\zeta \in \mathcal{H}$, then

$$\langle \dot{v}, \dot{w} \rangle_\zeta = -\frac{1}{4}(y^{-1/2} \dot{v} y^{-1/2})^* (y^{-1/2} \dot{w} y^{-1/2}),$$

   which is the Hilbertian product, in its version for $\mathcal{H}$.

Theorem 2 *The complex structure on $\mathcal{Q}_\rho$ is compatible with the Hilbertian product in the following sense:*

$$\langle \mathcal{J}_q \mathbf{X}, \mathbf{Y} \rangle_q = -\langle \mathbf{X}, \mathcal{J}_q \mathbf{Y} \rangle_q,$$

for $q = p_\mathbf{x} \in \mathcal{Q}_\rho$ and $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$.

Proof If $q = p_{\mathbf{x}}$ for $\mathbf{x} \in \mathcal{K}_\rho$, then $(2q-1)\mathbf{x} = \mathbf{x}$. Therefore, for $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$ it holds

$$\langle \mathbf{X}, \mathcal{J}_q \mathbf{Y} \rangle_q = -\Big[(\mathbf{x}, \theta_\rho(\mathbf{X}\mathbf{x}, \mathcal{J}_q \mathbf{Y}\mathbf{x}))\Big] = -\Big[(\mathbf{x}, i\mathbf{x}^*\mathbf{X}^*\rho\mathbf{Y}(2q-1)\mathbf{x})\Big] = i\langle \mathbf{X}, \mathbf{Y} \rangle_q .$$

Here $(2q-1)\mathbf{x} = \mathbf{x}$. On the other hand

$$\langle \mathcal{J}_q \mathbf{X}, \mathbf{Y} \rangle_q = -\Big[(\mathbf{x}, \theta_\rho(\mathcal{J}_q \mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big] = -\Big[(\mathbf{x}, i\theta_\rho(\mathbf{X}(2q-1)\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big] = -i\langle \mathbf{X}, \mathbf{Y} \rangle_q .$$

□

Remark 7 We saw earlier that the tangent space $(T\mathcal{Q}_\rho)_q$ is a right module over the corresponding fiber C_q of the coefficient bundle. The scalar field $\mathbb{C}$ lies naturally inside C_q as

$$\mathbb{C} \ni z \longleftrightarrow \Big[(\mathbf{x}, zI)\Big] \in C_q ,$$

because $uzIu^* = zI$ for all $u \in \mathcal{U}_\mathcal{A}$. Thus $(T\mathcal{Q}_\rho)_q$ is a $\mathbb{C}$-vector space. It is clear that the complex structure in $T\mathcal{Q}_\rho$ which we defined above (in terms of the identification Φ of (6)) coincides with the one induced by the operator $\mathcal{J}$.

Remark 8 One might wish to extend the argument above. Namely, to use the identification $\mathcal{Q}_\rho \simeq \mathcal{D} \subset \mathcal{A}$ in order to endow the tangent bundle $T\mathcal{Q}_\rho$ with a right action of the algebra $\mathcal{A}$. However, this action depends on the immersion (at the tangent level) and is not intrinsic. It works in the case of $\mathbb{C}$, as it works also for the center of $\mathcal{A}$: endomorphisms with matrices in the center of $\mathcal{A}$ have the same matrix for any basis of the given submodule.

2.3 The Symplectic Form ω

References for symplectic geometry are [5–8]. The differential forms that we consider here may have at least three types of coefficients, with respect to the scalar multiplying of the tangent vectors and also the values of the forms. The concept of form varies substantially according to the type the coefficient:

1. coefficients in C;
2. if $\mathcal{A}$ has a center valued trace, i.e., there exists a tracial linear map $\tau : \mathcal{A} \to \mathcal{B}$, with $\mathcal{B}$ a subalgebra of the center of $\mathcal{A}$, then the coefficients are the elements of $\mathcal{B}$;
3. if $\mathcal{A}$ has a scalar trace $\tau : \mathcal{A} \to \mathbb{C}$, then the coefficients are the complex numbers.

If $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$, the decomposition in selfadjoint and anti-selfadjoint parts

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = (\mathbf{X}, \mathbf{Y})_q + i\omega(\mathbf{X}, \mathbf{Y})_q$$

provides a generalized *Riemannian* form

$$(\mathbf{X}, \mathbf{Y})_q = Re\langle \mathbf{X}, \mathbf{Y}\rangle_q$$

and a generalized symplectic form

$$\omega(\mathbf{X}, \mathbf{Y})_q = Im\langle \mathbf{X}, \mathbf{Y}\rangle_q.$$

In this chapter we are interested in ω. Notice that ω is sesqui-linear and $\omega(\mathbf{Y}, \mathbf{X}) = -\omega(\mathbf{X}, \mathbf{Y})$.

We compute now the curvature of the canonical connection, which turns out to be a scalar multiple of the symplectic form ω.

To simplify the computation, we make the assumption that given $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$, we can construct a smooth map $q(t, s) \in \mathcal{Q}_\rho$ such that

$$q(0, 0) = q, \ \frac{\partial}{\partial t} q|_{(0,0)} = \mathbf{X} \ \text{ and } \ \frac{\partial}{\partial s} q|_{(0,0)} = \mathbf{Y},$$

and this map $q(t, s)$ is lifted to a map $\mathbf{x}(t, s) \in \mathcal{K}_\rho$. We assume also the existence of a cross section σ defined on a neighbourhood of q. $\sigma = \mathbf{x}a$, where both $\mathbf{x}$ and a are functions of (t, s). To abbreviate, we shall write with a dot $\dot{}$ the derivatives with respect to t, and with a tilde $'$ the derivatives with respect to s. Then, differentiating

$$D_{\mathbf{x}}\sigma = \mathbf{x}(\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a)$$

with respect to s, we get

$$\mathbf{x}\left(\dot{a}' + \theta_\rho(\mathbf{x}', \dot{\mathbf{x}})a + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}}'a + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a' + \theta_\rho(\mathbf{x}, \mathbf{x}')(\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a)\right) = D_{\mathbf{Y}} D_{\mathbf{x}}\sigma.$$

Interchanging $'$ with $\dot{}$ we get

$$\mathbf{x}\left(\dot{a}' + \theta_\rho(\dot{\mathbf{x}}, \mathbf{x}')a + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}}'a + \theta_\rho(\mathbf{x}, \mathbf{x}')\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})(a' + \theta_\rho(\mathbf{x}, \mathbf{x}')a)\right) = D_{\mathbf{x}} D_{\mathbf{Y}}\sigma.$$

This should be specialized at $(t, s) = (0, 0)$. Clearly, the lifting of $q(t, s)$ can be done in order that $\mathbf{x}'$ and $\dot{\mathbf{x}}$ are *horizontal* at $(0, 0)$ (i.e., that they belong to $N(q)$). Then we get

$$R(\mathbf{X}, \mathbf{Y})\sigma = D_{\mathbf{x}} D_{\mathbf{Y}}\sigma - D_{\mathbf{Y}} D_{\mathbf{x}}\sigma = \mathbf{x}(\theta_\rho(\dot{\mathbf{x}}, \mathbf{x}') - \theta_\rho(\mathbf{x}', \dot{\mathbf{x}}))a.$$

Since $\dot{\mathbf{x}}$ and $\mathbf{x}'$ are horizontal, we have that

$$\dot{\mathbf{x}} = \kappa_{\mathbf{x}}(\mathbf{X}) = \mathbf{X}\mathbf{x} \ \text{ and } \ \mathbf{x}' = \mathbf{Y}\mathbf{x}.$$

Then

$$R(\mathbf{X},\mathbf{Y})\sigma = \Big[(\mathbf{x}, \mathbf{x}(\theta_\rho(\mathbf{Xx},\mathbf{Yx}) - \theta_\rho(\mathbf{Yx},\mathbf{Xx}))a)\Big] = \Big[(\mathbf{x}, \mathbf{x}\theta_\rho(\mathbf{x},[\mathbf{X},\mathbf{Y}]\mathbf{x})a)\Big]. \tag{9}$$

Here we use that $\mathbf{X}$ and $\mathbf{Y}$, being tangent vectors of $\mathcal{Q}_\rho$, are θ_ρ-symmetric, i.e., $\mathbf{X}^*\rho = \rho\mathbf{X}$.

Note that $R(\mathbf{X},\mathbf{Y})$ is an endomorphism of $\mathcal{E}_q$. Recall the Hilbertian C-valued product in $\mathcal{Q}_\rho$

$$\langle \mathbf{X},\mathbf{Y}\rangle_q = -\Big[(\mathbf{x}, \theta_\rho(\mathbf{Xx},\mathbf{Yx})\Big]$$

for $\mathbf{x} \in \mathcal{K}_\rho$ such that $p_\mathbf{x} = q$, which is an element of C_q, i.e., an endomorphism of $\mathcal{E}_q$. Its imaginary part is

$$\begin{aligned} Im\langle \mathbf{X},\mathbf{Y}\rangle_q &= -\frac{i}{2}\Big[(\mathbf{x}, -\theta_\rho(\mathbf{Xx},\mathbf{Yx}) + \theta_\rho(\mathbf{Xx},\mathbf{Yx})^*)\Big] \\ &= -\frac{i}{2}\Big[(\mathbf{x}, \theta_\rho(\mathbf{Yx},\mathbf{Xx}) - \theta_\rho(\mathbf{Xx},\mathbf{Yx}))\Big] = \frac{i}{2}\Big[(\mathbf{x}, \theta_\rho(\mathbf{x},[\mathbf{X},\mathbf{Y}]\mathbf{x})\Big]. \end{aligned}$$

Thus, we have proved the following result, which is a kind of prequantization of $\mathcal{Q}_\rho$ [9]:

Theorem 3 *The curvature of the tautological bundle $\mathcal{E}$ and the Hilbertian product in $\mathcal{Q}_\rho$ are related by the following formula*

$$\frac{i}{2}\, R(\mathbf{X},\mathbf{Y})_q = Im\langle \mathbf{X},\mathbf{Y}\rangle_q = \omega(\mathbf{X},\mathbf{Y})_q.$$

2.4 The Moment Map

Consider a symplectic manifold M, and a Lie group G (with Lie algebra $\mathfrak{G}$) which acts on M by symplectomorphisms. In this situation a *moment map* is an equivariant function $\mu : M \to \mathfrak{G}^*$, the dual of $\mathfrak{G}$. Equivalently, μ can be given as a map $M \times \mathfrak{G} \to \mathbb{R}$. We shall adopt this latter point of view, and in our case the scalars will be replaced by the coefficient bundle C.

Recall from Remark 3 in Chap. 3 the Banach-Lie algebra $\mathfrak{U}_\rho$ of the Banach-Lie group $\mathcal{U}_\rho$

$$\mathfrak{U}_\rho = \mathfrak{U}_{0,\rho} \oplus \mathfrak{U}_{1,\rho},$$

where

$$\mathfrak{U}_{0,\rho} = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in M_2(\mathcal{A}) : a_1, a_2 \in \mathcal{A}_{ah}\} \tag{10}$$

and

$$\mathfrak{U}_{1,\rho} = \{\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A}) : a \in \mathcal{A}\}.$$

If $a \in \mathfrak{U}_\rho$ and $q \in \mathcal{Q}_\rho$, we put

$$f_a(q) = \frac{1}{2i}\Big[(\mathbf{x}, \theta_\rho(\mathbf{x}, a\mathbf{x}))\Big] = \frac{1}{2i}\Big[(\mathbf{x}, \mathbf{x}^*\rho a\mathbf{x})\Big], \tag{11}$$

for $\mathbf{x} \in \mathcal{K}_\rho$ such that $p_{\mathbf{x}} = q$. Again, a simple computation shows that if one chooses $\mathbf{x}u$ instead, the element $\theta_\rho(\mathbf{x}, a\mathbf{x})$ varies accordingly:

$$\theta_\rho(\mathbf{x}u, a\mathbf{x}u) = \langle \rho\mathbf{x}u, a\mathbf{x}u\rangle = u^*\langle \rho\mathbf{x}, a\mathbf{x}\rangle u = u^*\theta_\rho(\mathbf{x}, a\mathbf{x})u.$$

Thus, $f_a(q) \in C_q$. We call $f_a : \mathcal{Q}_\rho \to C$ *the moment map* of the generalized symplectic manifold $\mathcal{Q}_\rho$.

Proposition 3 *The moment map is equivariant with respect to the action of $\mathcal{U}_\rho$: if $c \in \mathcal{U}_\rho$, $q \in \mathcal{Q}_\rho$ and $a \in \mathfrak{U}_\rho$,*

$$f_{cac^{-1}}(cqc^{-1}) = f_a(q).$$

Proof Pick $\mathbf{x} \in \mathcal{K}_\rho$ such that $p_{\mathbf{x}} = q$. Recall that $c \in \mathcal{U}_\rho$ means that $\rho c^{-1} = c^*\rho$. Note that for $\mathbf{y} \in \mathcal{A}^2$,

$$cqc^{-1}\mathbf{y} = c(\mathbf{x}\theta_\rho(\mathbf{x}, c^{-1}\mathbf{y})) = c\mathbf{x}\theta_\rho(c\mathbf{x}, \mathbf{y}) = p_{c\mathbf{x}}\mathbf{y}.$$

Notice that, from Proposition 9, part 3 in Chap. 3, $c\mathbf{x} \in \mathcal{K}_\rho$ (see [10]; in general $a\mathbf{x} \in \mathcal{K}_\rho$ for any $a \in \mathcal{U}_\rho$, $\mathbf{x} \in \mathcal{K}_\rho$). Then

$$f_{cac^{-1}}(cqc^{-1}) = \Big[(c\mathbf{x}, \frac{1}{2i}\theta_\rho(c\mathbf{x}, cac^{-1}c\mathbf{x}))\Big] = \Big[(c\mathbf{x}, \frac{1}{2i}\theta_\rho(\mathbf{x}, a\mathbf{x}))\Big] = f_a(q).$$

□

Let us compute the covariant derivative of f_a. We use a horizontal lifting $\mathbf{x}(t) \in \mathcal{K}_\rho$ of the curve $q(t)$ in the direction of $\mathbf{X}$ at q (i.e., $q(0) = q$, $\dot{q}(0) = \mathbf{X}$ and $\dot{\mathbf{x}}(t) \in N(q(t))$). Then

$$D_{\mathbf{X}}f_a = \Big[(\mathbf{x}, \frac{1}{2i}\left(\theta_\rho(\dot{\mathbf{x}}, a\mathbf{x}) + \theta_\rho(\mathbf{x}, a\dot{\mathbf{x}})\right))\Big] = \Big[(\mathbf{x}, \frac{1}{2i}\left(\theta_\rho(\dot{\mathbf{x}}, a\mathbf{x}) - \theta_\rho(a\mathbf{x}, \dot{\mathbf{x}})\right))\Big]$$

$$= \Big[(\mathbf{x}, \frac{1}{2i}\left(\theta_\rho(\dot{\mathbf{x}}, a\mathbf{x}) - \theta_\rho(\dot{\mathbf{x}}, a\mathbf{x})^*\right))\Big] = \Big[(\mathbf{x}, \ Im\ \theta_\rho(\dot{\mathbf{x}}, a\mathbf{x}))\Big],$$

where we use that a is θ_ρ-anti-symmetric. Since $\dot{\mathbf{x}}$ is horizontal, it holds that $\dot{\mathbf{x}} = \mathbf{X}\mathbf{x}$. Then

$$D_{\mathbf{X}} f_a = \Big[(\mathbf{x},\ Im\ \theta_\rho(\mathbf{X}\mathbf{x}, a\mathbf{x})\Big]. \tag{12}$$

On the other hand, when computing the curvature of the canonical connection of $\mathcal{E}$, we proved that for any $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$

$$Im\ \langle \mathbf{X}, \mathbf{Y}\rangle_q = \Big[(\mathbf{x}, \frac{1}{2i}\left(\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}) - \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x})^*\right)\Big] = -\Big[(\mathbf{x}, Im\ \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big].$$

Now we consider for each $a \in \mathfrak{U}_\rho$ the vector field in $\mathcal{Q}_\rho$

$$\mathbf{X}_a(q) = (d\pi_q)_1(a) = [a, q],$$

where $\pi_q : \mathcal{U}_\rho \to \mathcal{Q}_\rho$ denotes the action map: $\pi_q(c) = cqc^{-1}$. Applying the formula above with $\mathbf{X} = \mathbf{X}_a$, we get

$$\omega(\mathbf{X}_a, \mathbf{Y})_q = Im\ \langle \mathbf{X}_a, \mathbf{Y}\rangle_q = \Big[(\mathbf{x}, -Im\ \theta_\rho(\mathbf{X}_a\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big].$$

Observe that $\mathbf{X}_a(q)\mathbf{x} = [a, q]\mathbf{x} = aq\mathbf{x} - qa\mathbf{x} = a\mathbf{x} - qa\mathbf{x}$ (recall that $q\mathbf{x} = p_{\mathbf{x}}\mathbf{x} = \mathbf{x}$). Note also that $\theta_\rho(qa\mathbf{x}, \mathbf{Y}\mathbf{x}) = 0$ because $\mathbf{Y}\mathbf{x} \in N(q)$. Thus $\theta_\rho(\mathbf{X}_a\mathbf{x}, \mathbf{Y}\mathbf{x}) = \theta_\rho(a\mathbf{x}, \mathbf{Y}\mathbf{x})$. Therefore

$$\omega(\mathbf{X}_a, \mathbf{Y})_q = \Big[(\mathbf{x}, -Im\ \theta_\rho(a\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big].$$

We proved before that $D_{\mathbf{Y}} f_a = \Big[\mathbf{x},\ Im\ \theta_\rho(\mathbf{Y}\mathbf{x}, a\mathbf{x}))\Big]$. Thus, we have shown that

$$D_{\mathbf{Y}} f_a = \omega(\mathbf{X}_a, \mathbf{Y}).$$

Suppose now that f is a section of C over some open subset $\mathcal{W}$ of $\mathcal{Q}_\rho$. Consider the covariant derivative Df as a 1-form on $\mathcal{W}$ with values in C: $Df(\mathbf{X}) = D_{\mathbf{X}} f$.

Definition 4 We say that a field $\mathbf{X}_f$ is the symplectic gradient of f if

$$D_{\mathbf{X}} f = \omega(\mathbf{X}, \mathbf{X}_f).$$

We are mainly interested in the case of functions of the form f_a, for $a \in \mathfrak{U}_\rho$,

$$f_a(q) = \frac{1}{2i}\Big[(\mathbf{x}, \theta_\rho(\mathbf{x}, a\mathbf{x}))\Big] = \frac{1}{2i}\Big[(\mathbf{x}, \mathbf{x}^*\rho a\mathbf{x})\Big],$$

where $q \in \mathcal{Q}_\rho$ and $\mathbf{x} \in \mathcal{K}_\rho$ satisfies $p_{\mathbf{x}} = q$. Recall that the Lie algebra $\mathfrak{U}_\rho$ consists of all matrices $a = \begin{pmatrix} a_{11} & \alpha \\ \alpha^* & a_{22} \end{pmatrix}$, where $a_{ii}^* = -a_{ii}$. Recall also the identity $D_{\mathbf{X}} f_a = \omega(\mathbf{X}, \mathbf{X}_a)$. This says that $\mathbf{X}_a$ is the symplectic gradient of f_a.

Recall also that $\mathbf{X}_a(q) = [a, q]$. Now given $a, b \in \mathfrak{U}_\rho$, we want to find the value of $\omega(\mathbf{X}_a, \mathbf{X}_b)$. at $q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and correspondingly $\mathbf{x} = \mathbf{e}_1 \in \mathcal{K}_\rho$. Since

$$X_a = [a, q] = \begin{pmatrix} 0 & -\alpha \\ \alpha^* & 0 \end{pmatrix}, \quad X_b = [b, q] = \begin{pmatrix} 0 & -\beta \\ \beta^* & 0 \end{pmatrix}$$

for some $\alpha, \beta \in \mathcal{A}$, then

$$\langle \mathbf{X}_a, \mathbf{X}_b \rangle_q = \theta_\rho\Big(\begin{pmatrix} 0 \\ \alpha^* \end{pmatrix}, \begin{pmatrix} 0 \\ \beta^* \end{pmatrix} \Big) = -\alpha\beta^*$$

and

$$\omega(\mathbf{X}_a, \mathbf{X}_b)_q = Im\ \langle \mathbf{X}_a, \mathbf{X}_b \rangle_q = \frac{1}{2i}(\beta\alpha^* - \alpha\beta^*).$$

We want to establish the relationship between $\omega(\mathbf{X}_a, \mathbf{X}_b)$ and $f_{[a,b]}$. To this effect, observe first that

$$\begin{aligned} \theta_\rho(\mathbf{e}_1, [a, b]\mathbf{e}_1) &= -\theta_\rho(a\mathbf{e}_1, b\mathbf{e}_1) + \theta_\rho(b\mathbf{e}_1, a\mathbf{e}_1) \\ &= -\theta_\rho\Big(\begin{pmatrix} a_{11} \\ \alpha^* \end{pmatrix}, \begin{pmatrix} b_{11} \\ \beta^* \end{pmatrix} \Big) + \theta_\rho\Big(\begin{pmatrix} b_{11} \\ \beta^* \end{pmatrix} \begin{pmatrix} a_{11} \\ \alpha^* \end{pmatrix} \Big) \\ &\alpha\beta^* - \beta\alpha^* + a_{11}b_{11} - b_{11}a_{11}. \end{aligned}$$

Then $\omega(\mathbf{X}_a, \mathbf{X}_b) = \frac{1}{2i}(\beta\alpha^* - \alpha\beta^*)$ and $f_{[a,b]} = \frac{1}{2i}(\alpha\beta^* - \beta\alpha^* + a_{11}b_{11} - b_{11}a_{11})$. So

$$\omega(\mathbf{X}_a, \mathbf{X}_b) = -f_{[a,b]} + \frac{1}{2i}(a_{11}b_{11} - b_{11}a_{11}).$$

Finally, $f_a(q) = \frac{1}{2i}\theta_\rho(\mathbf{e}_1, a\mathbf{e}_1) = \frac{1}{2i}a_{11}$, and

$$\omega(\mathbf{X}_a, \mathbf{X}_b) = -f_{[a,b]} + 2i[f_a, f_b]. \tag{13}$$

This equality holds at $q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, but a simple argument shows that (13) holds at every $q \in \mathcal{Q}_\rho$. Indeed, each of the terms involved is a function in $q \in \mathcal{Q}_\rho$ with $a, b \in \mathfrak{U}_\rho$ fixed. If we change q by cqc^{-1} for $c \in \mathcal{U}_\rho$, the values change by an inner automorphism. Finally, since $\mathcal{U}_\rho$ acts transitively on $\mathcal{Q}_\rho$, our claim is proven.

Given two sections f, g with their respective symplectic gradients $\mathbf{X}_f$ and $\mathbf{X}_g$, the *Poisson bracket* $\{f, g\}$ is defined as

$$\{f, g\} = \omega(\mathbf{X}_f, \mathbf{X}_g).$$

Then, we get

$$\{f_a, f_b\} = -f_{[a,b]} + 2i[f_a, f_b].$$

The term $2i[f_a, f_b]$ occurs because of the non-commutativity of the C*-algebras C_q, where the moment map takes its values.

Summarizing the facts of the last sections:

Theorem 4 *Consider the manifold $\mathcal{Q}_\rho$, with the tautological bundle $\mathcal{E}$ and the coefficient bundle $\mathcal{C}$. Then*

1. *There exist invariant connections in $\mathcal{E}$ and $\mathcal{C}$, linked by Leibnitz' rule (see (4)).*
2. *There exists a Hilbertian product in $T\mathcal{Q}_\rho$, with values in $\mathcal{C}$. This product is compatible with the right $\mathcal{C}$-module structure of $T\mathcal{Q}_\rho$.*
3. *The imaginary part ω of the Hilbertian product in $T\mathcal{Q}_\rho$ is the curvature of the tautological connection of $\mathcal{E}$. In this sense, the symplectic form ω is exact.*
4. *The map f_a ($a \in \mathfrak{U}_\rho$) is a moment map: the field X_a is the symplectic gradient of the function (cross section for $\mathcal{C}$) f_a. Here gradients are computed using the covariant derivative.*

2.5 The Invariant Finsler Structure

We proved in Sect. 3.1 of Chap. 3 that $\mathcal{Q}_\rho$ has a Finsler structure, which is invariant under $\mathcal{U}_\rho$. If $q \in \mathcal{Q}_\rho$ and $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$, define the norm

$$\|X\|_q = \||2q - I|^{-1/2}\mathbf{X}|2q - I|^{1/2}\|.$$

We recall that it is precisely this structure which, when translated to the disk $\mathcal{D}$, gives the Poincaré metric of the disk (Sect. 3.1 in Chap. 3). In this section we shall see that the Hilbertian product induces also in a natural way the same Finsler structure. To prove this fact, we must first indicate how to compute the norm of a given endomorphism $\varphi \in \mathcal{L}_{\mathcal{A}}(R(q))$.

Definition 5 Let $\varphi \in \mathcal{L}_{\mathcal{A}}(R(q))$. Then

$$|\varphi| := \sup_{0 \neq \mathbf{y} \in R(q)} \frac{\|\theta_\rho(\varphi(\mathbf{y}), \varphi(\mathbf{y}))\|^{1/2}}{\|\theta_\rho(\mathbf{y}, \mathbf{y})\|^{1/2}}.$$

Alternatively, θ_ρ is a C*-Hilbert module (positive) inner product in $R(q)$. Thus, the above formula is just the usual way to compute the norm of an endomorphism, when the module $R(q)$ is endowed with the C*-module norm.

We show that, in the presence of a basis $\mathbf{x} \in \mathcal{K}_\rho$ of $R(q)$ and a matrix $a \in \mathcal{A}$ for φ, the norm of the endomorphism is the norm of the matrix.

Lemma 2 *If $\varphi = [(\mathbf{x}, a)]$, then $|\varphi| = \|a\|$.*

Proof Let $\mathbf{y} = \mathbf{x}b \neq 0$ in $R(q)$. Then $\varphi(\mathbf{y}) = \mathbf{x}ab$. Thus

$$\frac{\|\theta_\rho(\varphi(\mathbf{y}), \varphi(\mathbf{y}))\|^{1/2}}{\|\theta_\rho(\mathbf{y}, \mathbf{y})\|^{1/2}} = \frac{\|\theta_\rho(\mathbf{x}ab, \mathbf{x}ab)\|^{1/2}}{\|\theta_\rho(\mathbf{x}b, \mathbf{x}b)\|^{1/2}} = \frac{\|b^*a^*ab\|^{1/2}}{\|b^*b\|^{1/2}} = \frac{\|ab\|}{\|b\|} \leq \|a\|.$$

On the other hand, taking $\mathbf{y} = \mathbf{x}$, one gets $\dfrac{\|\theta_\rho(\varphi(\mathbf{x}), \varphi(\mathbf{x}))\|^{1/2}}{\|\theta_\rho(\mathbf{x}, \mathbf{x})\|^{1/2}} = \|a\|$. □

Theorem 5 *Let $q \in \mathcal{Q}_\rho$ and $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$. Then, with the above Definition 5, one has*

$$|\langle \mathbf{X}, \mathbf{X} \rangle_q| = \|\mathbf{X}\|_q.$$

Proof Since both distributions of norms are $\mathcal{U}_\rho$-invariant, it suffices to consider the case $q = p = \frac{1}{2}(\rho + I) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. For the range of this projection we can choose the basis $\mathbf{e}_1$. If $\mathbf{X} \in (T\mathcal{Q}_\rho)_p$, then it is anti-Hermitian and co-diagonal, i.e., $\mathbf{X} = \begin{pmatrix} 0 & x_{12} \\ -x_{12}^* & 0 \end{pmatrix}$. Then

$$-\theta_\rho(\kappa_{\mathbf{e}_1}(\mathbf{X}), \kappa_{\mathbf{e}_1}(\mathbf{X})) = -\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & x_{12} \\ -x_{12}^* & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & x_{12} \\ -x_{12}^* & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_{12}x_{12}^*.$$

Therefore, by the above lemma, $|\langle \mathbf{X}, \mathbf{X} \rangle_p| = \|x_{12}x_{12}^*\|^{1/2} = \|x_{12}\|$.

On the other hand, if $q = p$, then clearly $\| \ \|_p$ is the usual norm in $M_2(\mathcal{A})$. Then

$$\|\mathbf{X}\|_p = \|\mathbf{X}\| = \|\mathbf{X}^*\mathbf{X}\|^{1/2} = \left\| \begin{pmatrix} 0 & x_{12} \\ -x_{12}^* & 0 \end{pmatrix}^2 \right\|^{1/2} = \left\| \begin{pmatrix} x_{12}x_{12}^* & 0 \\ 0 & x_{12}^*x_{12} \end{pmatrix} \right\|^{1/2}$$

$$= \max\{\|x_{12}x_{12}^*\|^{1/2}, \|x_{12}^*x_{12}\|^{1/2}\} = \|x_{12}\|.$$

□

2.6 The Scalar Case

In this section we shall consider the classical case when $\mathcal{A} = \mathbb{C}$. We shall check that the geometry induced by $\mathcal{U}_\rho$ on $\mathcal{D}$ is the classical hyperbolic geometry of the

Poincaré disk. In the scalar case, or more generally, when $\mathcal{A}$ is commutative, the coefficient bundle C consists of $\mathcal{A}$ in each fiber: the Hilbertian product, the moment map, and so forth, take values in $\mathcal{A}$. Indeed, if φ is an endomorphism of $R(q)$, with bases $\mathbf{x}$ and $\mathbf{x}u$, then the corresponding matrices of φ in these bases are

$$a \text{ and } uau^* = a,$$

respectively. That is, the coefficient bundle C ends up being $\mathcal{A}$.

In particular, in the case $\mathcal{A} = \mathbb{C}$, one obtains a complex structure for the unit disk $\mathcal{D}$. The goal of this section is to show that this structure is the classical complex structure of the Poincaré disk.

We first recall the isomorphism

$$\Phi_{\mathcal{D}} : \mathcal{D} \to \mathcal{Q}_\rho.$$

In the scalar case we get

$$\mathcal{D} \ni z \mapsto p_z = \frac{1}{1-|z|^2}\begin{pmatrix} 1 & -\bar{z} \\ z & -|z|^2 \end{pmatrix}.$$

The tangent spaces $(T\mathcal{D})_x$ identify with $\mathbb{C}$. Given $a \in \mathbb{C}$, regarded as a tangent vector in $T(\mathcal{D})_z$, let us denote by $\mathbf{X}_a = (d\Phi_{\mathcal{D}})_z(a)$ the corresponding tangent vector in $T(\mathcal{Q}_\rho)_{p_z}$. Clearly, one gets

$$\mathbf{X}_a = \frac{1}{(1-|z|^2)^2}\begin{pmatrix} \bar{a}z + a\bar{z} & -\bar{a} - a\bar{z}^2 \\ a + \bar{a}z^2 & -\bar{a}z - a\bar{z} \end{pmatrix}.$$

2.6.1 The Complex Inner Product

For the module $R(p_z)$, we chose the basis $\mathbf{x}_z = \dfrac{1}{(1-|z|^2)^{1/2}}\begin{pmatrix} 1 \\ z \end{pmatrix}$. Thus, the lifting $\kappa_{\mathbf{x}_z}(\mathbf{X}_a)$ is given by

$$\mathbf{X}_a\mathbf{x}_z = \frac{1}{(1-|z|^2)^{5/2}}\begin{pmatrix} \bar{a}z + a\bar{z} & -\bar{a} - a\bar{z}^2 \\ a + \bar{a}z^2 & -\bar{a}z - a\bar{z} \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix} = \frac{1}{(1-|z|^2)^{3/2}}\begin{pmatrix} a\bar{z} \\ a \end{pmatrix}.$$

Therefore, if $z \in \mathcal{D}$ and $a, b \in \mathbb{C}$ $(= (T\mathcal{D})_z)$

$$\langle a, b\rangle_z = -\theta_\rho(\mathbf{X}_a\mathbf{x}_z, \mathbf{X}_b\mathbf{x}_z) = -\frac{1}{(1-|z|^2)^{3/2}}\theta_\rho(\begin{pmatrix} a\bar{z} \\ a \end{pmatrix}, \begin{pmatrix} b\bar{z} \\ b \end{pmatrix}) = \frac{\bar{a}b}{(1-|z|^2)^2}, \tag{14}$$

which is the classical complex inner product in the Poincaré disk.

2.6.2 The Linear Connection

Let $\mathbf{X}$ be a tangent field defined on a neighbourhood of q in $\mathcal{Q}_\rho$, and $\mathbf{Y} \in (T\mathcal{Q}_\rho)_q$. Let $q(t)$ be a smooth curve adapted to $\mathbf{Y}$: $q(0) = q$ and $\dot{q}(0) = \mathbf{Y}$. Then the covariant derivative in $\mathcal{Q}_\rho$ is given by [11]

$$\nabla_{\mathbf{Y}} \mathbf{X}_q = \frac{d}{dt} \mathbf{X}_{q(t)}|_{t=0} + [\mathbf{X}_q, [\mathbf{Y}, q]].$$

Let now $a = a(z)$ be a $\mathbb{C}$-valued smooth map defined on a neighbourhood of $z_0 \in \mathcal{D}$, regarded as a tangent vector field in $\mathcal{D}$, and $b \in \mathbb{C}$ a tangent vector at z_0. To compute $\nabla_b a_{z_0}$, we have to compute $\nabla_{\mathbf{X}_b} \mathbf{X}_a$ at $z = z_0$, and identify this tangent vector as a matrix $\mathbf{X}_c$ for certain $c \in \mathbb{C}$ (at z!), and then $c = \nabla_b a_{z_0}$. In order to simplify this computation, we shall consider the case $z_0 = 0$. In fact, due to the invariance of the linear connection under the action of $\mathcal{U}_\rho$, this will suffice to identify the covariant derivative. In this case ($z_0 = 0$), we have

$$\mathbf{X}_a(z) = \frac{1}{(1-|z|^2)^2} \begin{pmatrix} \bar{a}(z)z + a(z)\bar{z} & -\bar{a}(z) - a(z)\bar{z}^2 \\ a(z) + \bar{a}(z)z^2 & -\bar{a}(z)z - a(z)\bar{z} \end{pmatrix}, \quad \mathbf{X}_b = \begin{pmatrix} 0 & -\bar{b} \\ b & 0 \end{pmatrix}.$$

Applying the formula above and choosing a smooth $z(t) \in \mathcal{D}$ such that $z(0) = 0$ and $\dot{z}(0) = b$ (for instance $z(t) = tb$), after straightforward computations one obtains

$$\nabla_{X_a}(X_b)_{E_1} = \begin{pmatrix} 0 & -\frac{\partial}{\partial b}\bar{a}(0) \\ \frac{\partial}{\partial b} a(0) & 0 \end{pmatrix} = X_{\frac{\partial}{\partial b} a(0)} \quad \text{at } z = 0.$$

Thus,

$$\nabla_b a_0 = \frac{\partial}{\partial b} a(0). \tag{15}$$

This coincides with the Levi-Civita connection of the classical metric of the Poincaré disk (at the origin).

2.6.3 The Moment Map

Note that the Lie algebra $\mathfrak{U}_\rho$ is given in this case by all matrices of the form

$$a = \begin{pmatrix} i\alpha & \omega \\ \bar{\omega} & i\beta \end{pmatrix} = i \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \ \omega \in \mathbb{C}.$$

Recall that if $q = p_{\mathbf{x}} \in \mathcal{Q}_\rho$ and $a \in \mathfrak{U}_\rho$, then the moment map is given by

$$f_a(q)) = \frac{1}{2i} \theta_\rho(\mathbf{x}, a\mathbf{x}).$$

If $z \in \mathcal{D}$ and $q = p_z$, we choose as above the basis $\dfrac{1}{(1-|z|^2)^{1/2}}\begin{pmatrix}1\\ z\end{pmatrix}$, and a is given as above, then

$$f_a(z) = f_a(q) = \frac{1}{1-|z|^2}\left\{\frac{1}{2}\theta_\rho(\begin{pmatrix}1\\ z\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1\\ z\end{pmatrix}) + \frac{1}{2i}\theta_\rho(\begin{pmatrix}1\\ z\end{pmatrix}, \begin{pmatrix}0 & \omega\\ \bar{\omega} & 0\end{pmatrix}\begin{pmatrix}1\\ z\end{pmatrix}\right\}$$

$$= \frac{1}{1-|z|^2}\left(\frac{1}{2}(\alpha - \beta|z|^2) + \frac{1}{2i}(\omega z - \bar{\omega}\bar{z})\right). \tag{16}$$

2.6.4 Commutative C*-Algebras

If $\mathcal{A}$ is commutative (i.e., $\mathcal{A} = C(\Omega, \mathbb{C})$ for some compact Hausdorf space Ω), the coefficient bundle also reduces to $\mathcal{A}$, as remarked at the beginning of this section. Moreover, it is clear that the computations done in this section can be carried over exactly in the same way. Thus, one gets a near classical situation, in which the Hilbertian product, the metric and the moment map take values in $\mathcal{A}$, and the formulas look the same as in the scalar case, replacing complex numbers by continuous functions.

2.7 Valuations

In this section we introduce valuation maps. Once a valuation map onto a commutative C*-algebra is chosen, the non commutative Kähler structure becomes a classical Kähler structure: measurements take values in a fixed scalar field, instead of being elements of the coefficient bundle $\mathcal{C}$. Most important, valuation maps will allow us to examine the convexity properties of the moment map.

In what follows, $\mathcal{F}$ denotes a commutative C*-algebra.

Definition 6 A valuation ν in $\mathcal{Q}_\rho$ is a differentiable map $\nu : \mathcal{C} \to \mathcal{F}$ with the following properties:

1. ν is positive in the following sense: for any $q \in \mathcal{Q}_\rho$, $\nu|_{C_q} : C_q \to \mathcal{F}$ is a positive linear map between C*-algebras. In particular, this implies that $\nu|_{C_q}$ is bounded.
2. ν is tracial: for any $q \in \mathcal{Q}_\rho$ and $a, b \in C_q$, $\nu(ab) = \nu(ba)$.
   Additionally, we say that ν is *faithful* if
3. for any $q \in \mathcal{Q}_\rho$ and $a \in C_q$, $\nu(a^*a) = 0$ implies $a = 0$.

Let us introduce the following examples, which show that the existence of ν is not an unlikely event.

Examples 6 1. If the base algebra $\mathcal{A}$ admits a trace τ with values in a commutative subalgebra $\mathcal{F} \subset \mathcal{A}$, then naturally a valuation ν is defined: $\nu(\varphi) = \tau(a)$, where a is the matrix of φ in $\mathbf{x}$, for any $q \in C_q$.

2. $\mathcal{A}$ need not admit a trace. Suppose that there exists a $*$-homomorphism onto a commutative algebra $\pi : \mathcal{A} \to \mathcal{F}$ (here $\mathcal{F}$ need not be a subalgebra of $\mathcal{A}$). In this case, $\pi(ab) = \pi(a)\pi(b) = \pi(b)\pi(a) = \pi(ba)$. One such example is the Toeplitz C*-algebra $\mathcal{T}(C(\mathbb{T})) = \{T_f : f \in C(\mathbb{T})\}$, where T_f denotes the Toeplitz operator with symbol f. $\mathcal{T}(C(\mathbb{T}))$ does not admit a trace, but it has a $*$-homomorphism onto a commutative algebra, namely

$$\pi : \mathcal{T}(C(\mathbb{T})) \to \mathcal{T}(C(\mathbb{T}))/\mathcal{K}(L^2(\mathbb{T})) \simeq C(\mathbb{T}).$$

Another example of this sort is the algebra

$$\mathcal{D} + \mathcal{K} = \{D + K : D \text{ diagonal and } K \text{ compact}\} \subset \mathcal{B}(\ell^2).$$

In this case, there is a homomorphism

$$\pi : \mathcal{D} + \mathcal{K} \to \mathcal{D} + \mathcal{K}/\mathcal{K}(\ell^2) \simeq \ell^\infty/c_0.$$

3. A third sort of example is obtained if $\mathcal{A}$ has a positive tracial map onto a commutative algebra.

Let us fix a valuation $\nu : C \to \mathcal{F}$.

The structures that we have defined in $\mathcal{Q}_\rho$, with values in C, have valuations which transform them in structures with values in $\mathcal{F}$. The main feature is the C-valued Hilbertian product: if $q = p_{\mathbf{x}}$,

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = -\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{y}}(\mathbf{Y})) = -\theta_\rho(\mathbf{Xx}, \mathbf{Yy}) \in C_q,$$

and applying ν

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q^\nu := -\nu\theta_\rho(\mathbf{X}, \mathbf{x}, \mathbf{Yy}). \tag{17}$$

This $\mathcal{F}$-valued inner product defines a Hermitian structure in $\mathcal{Q}_\rho$, where the scalar "field" is $\mathcal{F}$. Note that if ν is not faithful, then $\langle\ ,\ \rangle^\nu$ is positive semi-definite.

Next, we consider the connection, curvature and the symplectic form. We saw in Theorem 3 that the canonical connection in the tautological bundle $\xi \to \mathcal{Q}_\rho$ has curvature equal to ($-2i$-times) the imaginary part of the Hilbertian product (both terms in this assertion, considered as 2-forms in $\mathcal{Q}_\rho$ with values in C).

Let us consider now the following: pick a field of bases $\mathbf{x} \in \mathcal{K}_\rho$ defined on an open set and a cross section σ of ξ on this open set. We can write

$$D_{\mathbf{X}}\sigma = \mathbf{x}(\mathbf{X} \cdot a + \alpha(\mathbf{X})a)$$

where $\sigma = \mathbf{x}a$, a is an $\mathcal{A}$-valued function, $\mathbf{X} \cdot a$ is the directional derivative of a in the direction $\mathbf{X}$, and $\alpha(\mathbf{X})$ is an $\mathcal{A}$-valued 1-form in $\mathcal{Q}_\rho$ (what we called the 1-form of $\mathcal{Q}_\rho$ in the basis $\mathbf{x}$) . If we compute

$$D_{\mathbf{X}} D_{\mathbf{Y}} \sigma - D_{\mathbf{Y}} D_{\mathbf{X}} \sigma - D_{[\mathbf{X},\mathbf{Y}]} \sigma$$

for fields $\mathbf{X}, \mathbf{Y}$ in $(T\mathcal{Q}_\rho)_q$, we obtain the following expression for the curvature $R(\mathbf{X}, \mathbf{Y})\sigma$:

$$R(\mathbf{X}, \mathbf{Y})\sigma = \mathbf{x}((d\alpha)(\mathbf{X}, \mathbf{Y})a + [\alpha(\mathbf{X}), \alpha(\mathbf{Y})]a),$$

where $[\alpha(\mathbf{X}), \alpha(\mathbf{Y})] = \alpha(\mathbf{X})\alpha(\mathbf{Y}) - \alpha(\mathbf{Y})\alpha(\mathbf{X})$. On the other hand, we saw in Theorem 3 that

$$-\frac{1}{2i} R(\mathbf{X}, \mathbf{Y})_q = Im\ \langle \mathbf{X}, \mathbf{Y} \rangle_q,$$

where both terms are C-valued, and $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$. Applying the valuation ν we get

$$\nu(R(\mathbf{X}, \mathbf{Y})_q) = \nu(d\alpha(\mathbf{X}, \mathbf{Y}) + [\alpha(\mathbf{X}), \alpha(\mathbf{Y})]) = \nu d\alpha(\mathbf{X}, \mathbf{Y}).$$

Note that $\nu(\mathbf{X} \cdot \alpha(\mathbf{Y})) = \mathbf{X} \cdot d\alpha(\mathbf{Y})$, because ν is linear and bounded (it commutes with the derivatives). Hence

$$\nu\ d\alpha(\mathbf{X}, \mathbf{Y}) = \mathbf{X} \cdot \nu\alpha(\mathbf{Y}) - \mathbf{Y} \cdot \nu\alpha(\mathbf{X}) - \nu\alpha([\mathbf{X}, \mathbf{Y}]).$$

In other words: $\nu\ d\alpha(\mathbf{X}, \mathbf{Y}) = d\ \nu\alpha(\mathbf{X}, \mathbf{Y})$. Therefore

$$Im\ \langle \mathbf{X}, \mathbf{Y} \rangle_q^\nu = \nu\ Im \langle \mathbf{X}, \mathbf{Y} \rangle_q = -\frac{i}{2i} \nu R(\mathbf{X}, \mathbf{Y})_q = -\frac{1}{2i} \nu\ d\alpha(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2i} d\ \nu\alpha(\mathbf{X}, \mathbf{Y}).$$

Thus, recalling that $\omega(\mathbf{X}, \mathbf{Y}) = Im\ \langle \mathbf{X}, \mathbf{Y} \rangle$, we have that the alternate $\mathcal{F}$-valued 2-form $\nu\omega$ over $\mathcal{Q}_\rho$ satisfies the equality

$$\nu\omega(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2i} d\ \nu\alpha(\mathbf{X}, \mathbf{Y}). \tag{18}$$

We shall call $\nu\omega$ the *valuated symplectic form* over $\mathcal{Q}_\rho$, with values in $\mathcal{F}$. Then we have that the valuated symplectic form $\nu\omega$ is *exact*. Indeed, there exist global bases $\mathbf{x}$ defined in the whole $\mathcal{Q}_\rho$.

Recall the moment map defined in Sect. 2.4: for $a \in \mathfrak{U}_\rho$, $f_a : \mathcal{Q}_\rho \to C$, is defined by $f_a(q) = \frac{1}{2i}\Big[(\mathbf{x}, \mathbf{x}^* \rho a \mathbf{x})\Big]$. Recall also the equality $D_{\mathbf{X}} f_a = \omega(\mathbf{X}, \mathbf{X}_a)$. Let us apply the valuation to the moment map

$$f_a^\nu = \nu f_a : \mathcal{Q}_\rho \to \mathcal{F}.$$

Note that

$$\mathbf{X} \cdot f_a^\nu = X \cdot \nu f_a = \nu D_{\mathbf{X}} \cdot f_a,$$

Therefore $\mathbf{X} \cdot f_a^\nu = \nu\omega(\mathbf{X}, \mathbf{X}_a)$. This means that f^ν is a moment map. Let us understand now the map f^ν as a map from $\mathcal{Q}_\rho$ to the dual of $\mathfrak{U}_\rho$. Here "dual" means the

space of bounded linear "functionals" with values in $\mathcal{F}$. To do this, we shall consider the following tracial functional :

$$\tau : M_2(\mathcal{A}) \to \mathcal{F}\ , \quad \tau \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{2}\nu(a_{11} + a_{22}).$$

Apparently, τ is positive, $\tau(I) = 1$ and verifies $\tau(ab) = \tau(ba)$. Then, if $q = p_{\mathbf{x}}$, we have

$$f_a^{\nu}(q) = \nu\theta_\rho(\mathbf{x}, a\mathbf{x}) = \nu(\mathbf{x}^*\rho a\mathbf{x}) = \tau(\mathbf{x}^*\rho a\mathbf{x}),$$

where the last equality uses the fact that, on elements a of $\mathcal{A}$ (regarded as scalar matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$), ν coincides with τ. Then, using the trace property of τ, $\tau(\mathbf{x}^*\rho a\mathbf{x}) = \tau(\mathbf{x}\mathbf{x}^*\rho a) = \tau(qa)$, i.e.,

$$f_a^{\nu}(q) = \tau(qa). \tag{19}$$

This formula allows one to clearly identify (by means of ν) the moment map as a map from $\mathcal{Q}_\rho$ with values in the tangent space of the Lie algebra $\mathfrak{U}_\rho$ of the group $\mathcal{U}_\rho$: to $q \in \mathcal{Q}_\rho$ corresponds the $\mathcal{F}$-valued linear functional $\tau(q\ \cdot)$, with density matrix q.

Remark 9 The dual space considered is subordinated to the valuation ν. It consists of bounded linear functionals defined on $\mathfrak{U}_\rho$, with values in $\mathcal{F}$. If, additionally, the algebra of scalars $\mathcal{F}$ in which one chooses to take measurements is a subalgebra of $\mathcal{A}$, then these functionals $\tau(q\ \cdot)$ are also $\mathcal{F}$-linear.

Next we shall discuss a property of the moment map, which mimics the theorem of [12–15] on compact symplectic manifolds acted by a torus. We shall consider therefore a subgroup of the full group $\mathcal{U}_\rho$ acting in $\mathcal{Q}_\rho$, namely the diagonal group

$$\mathbf{D}_\rho = \{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} : u_i \in \mathcal{U}_\mathcal{A}\} \subset \mathcal{U}_\rho,$$

which will play the role of a torus. Note that $\mathbf{D}_\rho = \mathcal{U}_\rho \cap \mathcal{U}_2(\mathcal{A})$, i.e., the unitary matrices in $M_2(\mathcal{A})$ which preserve the form θ_ρ.

In classical symplectic geometry, the restriction of the action to the subgroup induces a restricted moment map, which, when regarded as a map from the manifold to the dual of the Lie algebra of the acting group, consists in composing the moment of the full group with the projection of the Lie algebra of the full group onto the Lie algebra of the subgroup.

In our case, we shall restrict the action to $\mathbf{D}_\rho$. Note that the Lie algebra of $\mathbf{D}_\rho$ is the subalgebra $\mathfrak{U}_{0,\rho}$ given in (10)

$$\mathfrak{U}_{0,\rho} = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_i^* = -a_i\}.$$

The projection is given by

$$\mathfrak{U}_\rho \to \mathfrak{U}_{0,\rho}\ , \quad \begin{pmatrix} a_1 & b \\ b^* & a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

The following is an Atiyah-Guillemin-Sternberg's type of convexity result.

Theorem 7 *The image of moment map f^ν of the restricted action of the group $\mathbf{D}_\rho$,*

$$\mathcal{Q}_\rho \ni q \stackrel{f^\nu}{\longmapsto} \tau(q\ \cdot) \in \mathcal{L}(\mathfrak{U}_{0,\rho}, \mathcal{F}),$$

regarded as a subset of the space $\mathcal{L}(\mathfrak{U}_{0,\rho}, \mathcal{F})$ of linear functionals $\mathfrak{U}_{0,\rho} \to \mathcal{F}$, is a convex set.

Proof As done previously in Sect. 2.1, we shall use the model $\mathcal{D} \simeq \mathcal{Q}_\rho$, to prove our statement. Recall that each $q = p_z$, for some $z \in \mathcal{D}$, where $p_z = (1 - zz^*)^{-1} \begin{pmatrix} 1 & -z^* \\ z & -zz^* \end{pmatrix}$. Then, if $a_0 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \mathfrak{U}_{0,\rho}$, we get

$$f_a(p_z) = \tau\Big((1 - zz^*)^{-1} \begin{pmatrix} 1 & -z^* \\ z & -zz^* \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\Big) = \nu((1 - zz^*)^{-1}(a_1 - zz^* a_2))$$

$$= \nu\Big(\begin{pmatrix} (1 - zz^*)^{-1} & -(1 - zz^*)^{-1} zz^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\Big).$$

Therefore, if the elements of $\mathfrak{U}_{0,\rho}$ are represented as pairs $(a_1, a_2) \in \mathcal{A}^2_{ah}$, via the valuation ν, the image of the moment map of the group $\mathbf{D}_\rho$ identifies with the set of pairs

$$\{(c_1, c_2) \in \mathcal{A}^2 : c_1 \in G^+, c_1 + c_2 = 1\},$$

which is clearly a convex set. □

2.8 *The Liouville* 1*-Form*

We begin this section with the explicit computation of the lifting form introduced in Sect. 3.3.1 of Chap. 3, for the case of $\mathcal{H}$.

2.8.1 The Lifting Form of $\mathcal{H}$

Pick $\mathbf{x} \in \mathcal{K}_{\rho'}$, which means that $2\ Im\ x_1^* x_2 = 1$. Note the fact that $x_1 \in G_{\mathcal{A}}$ implies that also $x_2 \in G_{\mathcal{A}}$. As in the model $\mathcal{D}$, every $\mathbf{x} \in \mathcal{K}_{\rho'}$ gives rise to a projection (its representative in $\mathcal{Q}_{\rho'}$): $p_{\mathbf{x}} = \mathbf{x}\mathbf{x}^* \rho_H$. Consider its complement

$$1 - p_{\mathbf{x}} = \begin{pmatrix} 1 - ix_1x_2^* & ix_1x_1^* \\ -ix_2x_1^* & 1 + ix_2x_1^* \end{pmatrix}.$$

Note the second column $\begin{pmatrix} ix_1x_1^* \\ 1 + ix_2x_1^* \end{pmatrix} = \begin{pmatrix} ix_1 \\ (x_1^*)^{-1} + ix_2 \end{pmatrix} x_1^*$, which means that

$$\begin{pmatrix} ix_1 \\ (x_1^*)^{-1} + ix_2 \end{pmatrix}$$

generates the nullspace of $p_{\mathbf{x}}$. Let us denote this vector by

$$\mathbf{x}_\perp := \begin{pmatrix} ix_1 \\ (x_1^*)^{-1} + ix_2 \end{pmatrix}. \tag{20}$$

Note that $\theta_{\rho'}(\mathbf{x}_\perp, \mathbf{x}_\perp) = -1$. The pair $\{\mathbf{x}, \mathbf{x}_\perp\}$ forms a $\theta_{\rho'}$-orthogonal basis for $\mathcal{A}^2$. Any element $\mathbf{z} \in A^2$ is written

$$\mathbf{z} = \mathbf{x}\theta_{\rho'}(\mathbf{x}, \mathbf{z}) - \mathbf{x}_\perp\theta_{\rho'}(\mathbf{x}_\perp, \mathbf{z}).$$

Note also that $\mathbf{x}_\perp = \mathbf{x}i + \mathbf{e}_2(x_1^*)^{-1}$, which gives $\mathbf{x}_\perp = \mathbf{x} - \mathbf{e}_2 i(x_1^*)^{-1}$.

The map $\mathcal{K}_{\rho'} \to \mathcal{H}$, $\mathbf{x} \mapsto \zeta = x_2x_1^{-1}$ is the $\mathcal{H}$ valued version of the projection map $\mathbf{x} \to p_{\mathbf{x}}$. Its tangent map at $\mathbf{x}$, sends $\mathbf{v} \in (T\mathcal{K}_{\rho'})_{\mathbf{x}}$ to

$$\mathbf{v} \mapsto \dot{v} := v_2x_1^{-1} - x_2x_1^{-1}v_1x_1^{-1}.$$

Suppose now that $\mathbf{v} \in N(p_{\mathbf{x}})$: $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 - i(x_1^*)^{-1} \end{pmatrix} \lambda$, so that

$$\dot{v} = (x_2 - i(x_1^*)^{-1})\lambda x_1^{-1} - x_2x_1^{-1}x_1\lambda x_1^{-1} = (x_2\lambda - i(x_1^*)^{-1}\lambda - x_2\lambda)x_1^{-1} = -i)(x_1^*)^{-1}\lambda x_1^{-1},$$

and thus $\lambda = i\ x_1^*\dot{v}x_1$. Then, given $\mathbf{x}$ in $\mathcal{K}_{\rho'}$ which projects over ζ in $\mathcal{H}$, and $\dot{v} \in (T\mathcal{H})_\zeta$, we have the (lifting) form, as in Sect. 3.3.1 of Chap. 3

$$\kappa_{\mathbf{x}}^H(\dot{v}) = \begin{pmatrix} x_1 \\ x_2 - i\ x_1(x_1^*)^{-1} \end{pmatrix} ix_1^*\dot{v}x_1 = \begin{pmatrix} x_1x_1^* \\ x_2x_1^* - i \end{pmatrix} i\ \dot{v}x_1.$$

Or equivalently

$$\kappa_{\mathbf{x}}^H(\dot{v}) = (\mathbf{x}\ i - \mathbf{e}_2 i\ (x_1^*)^{-1})i\ x_1^*\dot{v}x_1 = (-\mathbf{x}x_1^* + \mathbf{e}_2)\dot{v}x_1.$$

There is a natural global cross section for the map $\mathcal{K}_{\rho'} \ni \mathbf{x} \mapsto \zeta \in \mathcal{H}$, namely

$$\zeta = x + iy \mapsto \begin{pmatrix} 1 \\ \zeta \end{pmatrix} (2y)^{-1/2}.$$

Then we have

$$\kappa_{\mathbf{x}}^{H}(\dot{v}) = \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - i \end{pmatrix} i\ \dot{v}(2y)^{-1/2}. \tag{21}$$

We want to compute the following product in $\mathcal{H}$.

Consider the expressions (21) for two tangent vectors $\dot{v}, \dot{w}$ in $(T\mathcal{H})_\zeta$:

$$\mathbf{v} = \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - i \end{pmatrix} i\ \dot{v}(2y)^{-1/2}\ , \quad \mathbf{w} = \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - i \end{pmatrix} i\ \dot{w}(2y)^{-1/2}.$$

Then we have

$$\theta_{\rho'}(\mathbf{v}, \mathbf{w}) = \langle \dot{v}, \dot{w} \rangle = (2y)^{-1/2} \dot{v}^* \theta_{\rho'}\Big(\begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - 1 \end{pmatrix}, \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - 1 \end{pmatrix} \Big) \dot{w}(2y)^{-1/2}$$

$$= -(2y)^{-1/2} \dot{v}^* (2y)^{-1} \dot{w}(2y)^{-1/2} = -\frac{1}{4}(y^{-1/2}\dot{v}y^{-1/2})^*(y^{-1/2}\dot{w}y^{-1/2}).$$

Suppose now that the algebra has a trace τ onto a commutative subalgebra $\mathcal{B} \subset \mathcal{A}$. If we take the trace of the above expression, we get

$$\tau\langle \dot{v}, \dot{w} \rangle = -\tau((2y)^{-1}\dot{v}^*(2y)^{-1}\dot{w}). \tag{22}$$

Here $y^{-1/2}\dot{v}y^{-1/2}$ is the translation to the point $y^{-1/2}\zeta y^{-1/2}$ of the vector $\dot{v}$. Note that $y^{-1/2}\zeta y^{-1/2} = y^{-1/2}xy^{-1/2} + i$.

Therefore $\langle \dot{v}, \dot{w} \rangle_\zeta = \langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0}$, where $\zeta_0 = y^{-1/2}xy^{-1/2} + i$. When the imaginary part equals 1, the inner product has the simpler expression $\langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0} = -\frac{1}{4}\dot{v}_0^*\dot{w}_0$. Or, explicitly, if $\dot{v}_0 = \dot{x} + i\ \dot{y}$ and $\dot{w}_0 = \dot{\xi} + i\ \dot{\eta}$, then $\dot{v}_0^*\dot{w}_0 = (\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + i(\dot{x}\dot{\eta} - \dot{y}\dot{\xi})$, and

$$\tau\langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0} = -\frac{1}{4}\{\tau(\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + i\ \tau(\dot{x}\dot{\eta} - \dot{y}\dot{\xi})\}.$$

Both traces in the above expression are selfadjoint elements of $\mathcal{B}$ (or real numbers if the trace is numerical). The imaginary part of $\tau\langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0}$ is essentially the (trace of the) symplectic form ω. This expression corresponds strictly to $dp \wedge dq$ in the classical setting, where p varies in the imaginary part and q in the real part. This shows that $\mathcal{H} = TG^+$ as a true dynamical system.

Let us write down the real and imaginary part of $\tau\langle\ ,\ \rangle$ at any point $\zeta \in \mathcal{H}$ (non necessarily with $Im\ \zeta = 1$). With the current notation, we get

$$\tau\langle \dot{v}, \dot{w} \rangle_\zeta = -\frac{1}{4}\{\tau(\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + i\ \tau(\dot{x}\dot{\eta} - \dot{y}\dot{\xi})\}, \tag{23}$$

where $\dot{x} + i\ \dot{y} = y^{-1/2}\dot{v}y^{-1/2}$ and $\dot{\xi} + i\ \dot{\eta} = y^{-1/2}\dot{w}y^{-1/2}$.

Let us compute now the 1-*form of Liouville* which induces the symplectic form ω of $\mathcal{H}$. The 1-form of Liouville is a form on the cotangent bundle of the manifold. Here

it shall be presented as a 1-form in the *tangent* bundle $TG^+ \simeq \mathcal{H}$, identifying the tangent bundle TG^+ with the co-tangent bundle T^*G^+. To perform this identification, we shall use the trace τ, and the action of $G_{\mathcal{A}}$ on G^+. Given a vector $\dot{v} \in (T\mathcal{H})_\zeta$, for $\zeta = x + i\ y \in \mathcal{H}$, the Liouville form α maps ζ into an element of $\mathcal{B}$. We must project $\dot{v}$ from its tangency point ζ to the point $y \in G^+$, having in mind that $\zeta = x + i\ y$ represents the tangent x at the point y. Finally, we evaluate x in $\dot{v}$. To perform this task, we translate to $y = 1$, and compute

$$\alpha_y(\dot{v}) = \tau(y^{-1/2}xy^{-1/2}y^{-1/2}\dot{v}y^{-1/2}) = \tau(y^{-1}xy^{-1}\dot{v}). \tag{24}$$

This is the Liouville 1-form.

Now we differentiate this 1-form in $\mathcal{H}$, i.e., we compute $d\alpha(\dot{v}, v') = \dot{v}\alpha(v') - v'\alpha(\dot{v}) - \alpha([\dot{v}, v'])$. Consider a function in the variables s, t whose derivatives produce the fields $\dot{v}$, v' when differentiated with respect to t and s, respectively. We must compute

$$\frac{\partial}{\partial t}\alpha_\zeta(v') - \frac{\partial}{\partial s}\alpha_\zeta(\dot{v}),$$

because there is no need to subtract $\alpha([\dot{v}, v'])$, since it is trivial. Then

$$\frac{\partial}{\partial t}\alpha_\zeta(v') = \tau(-y^{-1}\dot{v}y^{-1}xy^{-1}v' + y^{-1}\dot{x}y^{-1}y' - y^{-1}xy^{-1}\dot{y}y^{-1}y' + y^{-1}xy^{-1}\dot{y}')$$

and

$$\frac{\partial}{\partial s}\alpha_\zeta(\dot{v}) = \tau(-y^{-1}y'y^{-1}xy^{-1}\dot{y} + y^{-1}x'y^{-1}\dot{y} - y^{-1}xy^{-1}y'y^{-1}\dot{y} + y^{-1}xy^{-1}y').$$

Therefore

$$(d\alpha)_\zeta(\dot{v}, v') = \frac{\partial}{\partial t}\alpha_\zeta(v') - \frac{\partial}{\partial s}\alpha_\zeta(\dot{v}) = \tau(y^{-1}\dot{x}y^{-1}y' - y^{-1}x'y^{-1}\dot{y}).$$

Now, comparing this last expression with the imaginary part of the trace of the Hilbertian product form, and adapting the notation of both computations, we get.

Theorem 8 *The symplectic form ω is* exact*:*

$$(d\alpha)_\zeta(\dot{v}, \dot{w}) = \omega(\dot{v}, \dot{w}).$$

Remark 10 Since G^+ can be regarded as a submanifold of $\mathcal{H}$:

$$G^+ \ni y \mapsto iy \in \mathcal{H},$$

then we can compute the Hilbertian product of $\mathcal{H}$ on this submanifold. This coincides with the Riemannian structure of G^+. If $x_1, x_2 \in (TG^+)_y$, we have that $\zeta_1 = x_1 + i\ y$ and $\zeta_2 = x_2 + i\ y$ belong to $\mathcal{H}$. Define

$$[x_1, x_2]_y = \tau((y^{-1/2}x_1y^{-1/2})(y^{-1/2}x_2y^{-1/2})) = \tau(y^{-1}x_1y^{-1}x_2).$$

It is a $\mathcal{B}$-valued, bilinear (it takes selfadjoint values) and positive semidefinite form. Note also that this form is invariant under the action of G on G^+. In fact, $g \cdot y = (g^{-1})^* y g^{-1}$, for $g \in G$ and $y \in G^+$. The action defines a linear isomorphism in $\mathcal{A}$, so that the same formula gives the induced action in the tangent spaces of G^+: $g \cdot x = (g^{-1})^* x g$, if $x \in (TG^+)_y$. A straightforward computation shows that

$$[g \cdot x_1, g \cdot x_2]_{g\cdot y} = [x_1, x_2]_y.$$

If $\mathcal{A} = M_n(\mathbb{C})$, $\mathcal{B} = \mathbb{C}$ and τ is the usual trace for $n \times n$ matrices, this inner product is the usual Riemannian metric for the homogeneous space of positive definite $n \times n$ matrices.

On the other hand, if we embed $G^+ \hookrightarrow \mathcal{H}$, by means of $y \mapsto i\ y$, i.e., we regard G^+ as the imaginary positive axis of $\mathcal{H}$, then for $x_1, x_2 \in (TG^+)_y$ it holds

$$\begin{aligned}\langle i\ x_1, i\ x_2\rangle_{i\ y} &= -\frac{1}{4}(y^{-1/2} i\ x_1 y^{-1/2})^*(y^{-1/2} i\ x_2 y^{-1/2}) \\ &= -\frac{1}{4}(y^{-1/2}x_1y^{-1/2})(y^{-1/2}x_2y^{-1/2}),\end{aligned}$$

and, thus,

$$\tau\langle i\ x_1, i\ x_2\rangle_{i\ y} = -\frac{1}{4}[x_1, x_2]_y.$$

This means that the trace of the Hilbertian product, restricted to the positive imaginary axis in $\mathcal{H}$, gives, essentially, the Riemannian metric of the space G^+.

A geometric case of an algebra with a central trace is the following. Consider a complex vector bundle $E \to B$ with compact base space B, endowed with a Riemannian metric $\langle e, e'\rangle_b, b \in B, e, e' \in E_b$ (the fiber of E over b). Consider the fiber bundle $\mathrm{End}(E) \to B$ of endomorphisms of the vector bundle E, and let $\mathcal{A}$ be the algebra $\Gamma(\mathrm{End}(E))$ of the continuous global cross sections of $\mathrm{End}(E)$. Since each E_b is a (finite dimensional) Hilbert space, $\mathrm{End}(E_b)$ is a C*-algebra. The space $\Gamma(\mathrm{End}(E))$ of cross sections has therefore the norm $\|\varphi\| = \sup_{b\in B} \|\varphi_b\|$, where $\varphi_b : E_b \to E_b$ and $\|\varphi_b\|$ is the usual norm of linear operators. With this norm, $\mathcal{A}$ is a C*-algebra. The center $Z(\mathcal{A})$ of this algebra is the space of scalar sections λ in $\mathrm{End}(E)$ (homotetic in each fiber). The central trace is given by $\mathbf{tr} : \mathcal{A} \to Z(\mathcal{A})$, $\mathbf{tr}(\sigma)_b = Tr(\sigma_b)$, $b \in B$, with Tr the usual trace of E_b. More specifically, B could be a compact manifold, and E the complexification of its tangent bundle, with an Hermitian metric. This case is interesting due to the equivalence, as homogeneous spaces, of the disk $\mathcal{D}$ and the Poincaré halfspace $\mathcal{H}$ of the algebra $\mathcal{A}$. This homogeneous space can be thought as the tangent bundle TG^+ of the space G^+ of positive and invertible elements of $\mathcal{A}$. In this context, an element of $\mathcal{H}$ is a pair (X, a) with $a \in G^+$ and $X \in (TG^+)_a$. The element $a \in G^+$ represents a Riemannian metric in B, and a possible tangent vector

X (a selfadjoint element of $\mathcal{A}$) could be the Ricci curvature of the metric a. In this manner, the geometry of $\mathcal{H}$ is linked to the deformation of the pairs (*Riemannian metric, Ricci curvature*), viewed as elements of TG^+.

3 Projective Geometry

3.1 The Hyperbolic Part of $\mathcal{A}\mathbb{P}_1$

We say that an element $\mathbf{x} \in \mathcal{A}^2$ is *regular* if $\mathbf{x} \in Gl_2(\mathcal{A})\mathbf{e_1}$, i.e., if $\mathbf{x}$ is the first column of a matrix in $Gl_2(\mathcal{A})$. A *regular rank one submodule* of $\mathcal{A}^2$ is a right $\mathcal{A}$-module $\ell \subset \mathcal{A}^2$ of the form $\ell = [\mathbf{x}] := \{\mathbf{x}a;\, a \in \mathcal{A}\}$ for some regular $\mathbf{x} \in \mathcal{A}^2$.

Definition 7 The projective line of $\mathcal{A}$ is the set $\mathcal{A}\mathbb{P}_1$ of all regular rank one submodules of $\mathcal{A}^2$.

We define the *cross ratio* $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ of four submodules $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathcal{A}\mathbb{P}_1$, following ideas by Zelikin [16].

Definition 8 We denote by $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ the (possibly empty) set of module homomorphisms $\varphi : \ell_3 \to \ell_3$ of the form $\varphi = \psi\eta$, where the homomorphisms $\eta : \ell_3 \to \ell_2$, $\psi : \ell_2 \to \ell_3$ satisfy $x - \psi(x) \in \ell_4$ and $y - \eta(y) \in \ell_1$, for all $x \in \ell_2$, $y \in \ell_3$.

This set may contain more than one element. In the next section, we discuss existence and uniqueness of elements in $CR(\ell_1, \ell_2, \ell_3, \ell_4)$.

Consider the subset $\mathcal{A}\mathbb{P}_1^\rho$ of $\mathcal{A}\mathbb{P}_1$, which we call the *hyperbolic part* of $\mathcal{A}\mathbb{P}_1$:

$$\mathcal{A}\mathbb{P}_1^\rho = \{[\mathbf{x}] : \mathbf{x} \in \mathcal{K}_\rho\}.$$

Note that the elements of $\mathcal{K}_\rho$ are regular because they are the first columns of matrices in $\mathcal{U}_\rho$. It is easy to see that $\mathcal{A}\mathbb{P}_1^\rho$ is properly contained in $\mathcal{A}\mathbb{P}_1$.

Proposition 4 *The group $\mathcal{U}_\rho$ acts transitively on $\mathcal{A}\mathbb{P}_1^\rho$.*

Proof For $a \in \mathcal{U}_\rho$ and $[\mathbf{x}] \in \mathcal{A}\mathbb{P}_1^\rho$, define

$$a \cdot [\mathbf{x}] = [a\mathbf{x}].$$

This operation is well defined: if $\mathbf{x}, \mathbf{x}'$ span the same submodule then $\mathbf{x}' = \mathbf{x}u$ for some unitary $u \in \mathcal{A}$. Then $a\mathbf{x}' = (a\mathbf{x})u$ and $[a\mathbf{x}'] = [a\mathbf{x}]$. To finish the proof, it suffices to recall that $\mathcal{U}_\rho$ acts transitively on $\mathcal{K}_\rho$ (Proposition 9, part 5 in Chap. 3). □

There exist many natural ways of endowing $\mathcal{A}\mathbb{P}_1$ with a differentiable structure, see for instance [17, 18]. Here we choose to define the differentiable structure only on $\mathcal{A}\mathbb{P}_1^\rho$. We have shown that $\mathcal{Q}_\rho = \{p_\mathbf{x} : \mathbf{x} \in \mathcal{K}_\rho\}$, where $p_\mathbf{x}$ is the unique θ_ρ-orthogonal projection with range $[\mathbf{x}]$.

Proposition 5 *The mapping* $\mathbf{R} : \mathcal{Q}_\rho \to \mathcal{AP}_1^\rho$ *defined by* $\mathbf{R}(q) = R(q)$ *is a bijection, with* $\mathbf{R}^{-1}([\mathbf{x}]) = p_{\mathbf{x}}$. *Moreover,* $\mathbf{R}(aqa^{-1}) = a \cdot \mathbf{R}(q)$.

Proof Straightforward computation. □

The natural bijection $\mathbf{R}$ allows one to transfer the differentiable structure of $\mathcal{Q}_\rho$ to $\mathcal{AP}_1^\rho$. For instance, the tangent space at $[\mathbf{x}] \in \mathcal{AP}_1^\rho$, for $\mathbf{x} \in \mathcal{K}_\rho$, is identified as

$$(T\mathcal{AP}_1^\rho)_{[\mathbf{x}]} = \{X \in M_2(\mathcal{A}) : X^\sharp = X,\ p_{\mathbf{x}}X = X(1 - p_{\mathbf{x}})\}$$

$$\simeq \{[\mathbf{y}] : \mathbf{y} \in \mathcal{A}^2, \theta_\rho(\mathbf{x}, \mathbf{y}) = 0, \theta_\rho(\mathbf{y}, \mathbf{y}) = 1\} = N(p_{\mathbf{x}}).$$

The isomorphism implementing $\simeq$ above is the map $X \mapsto X\mathbf{x}$, from $(T\mathcal{AP}_1^\rho)_{[\mathbf{x}]}$ onto $N(p_{\mathbf{x}})$.

If $\ell \in \mathcal{AP}_1^\rho$, denote by

$$\ell^{\perp_\rho} = \{\mathbf{y} \in \mathcal{A}^2 : \theta_\rho(\mathbf{y}, \mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \ell\}.$$

Note that $(T\mathcal{AP}_1^\rho)_{[\mathbf{x}]} \simeq [\mathbf{x}]^{\perp_\rho}$.

3.2 The Invariant Metric in $\mathcal{AP}_1^\rho$

Given $\ell \in \mathcal{AP}_1^\rho$ and $V \in (T\mathcal{AP}_1^\rho)_\ell$, fix a generator $\mathbf{x}_0 \in \mathcal{K}_\rho$ for ℓ, i.e., $[\mathbf{x}_0] = \ell$ and $\theta_\rho(\mathbf{x}_0, \mathbf{x}_0) = 1$. Recall that $\mathbf{x}_0$ is determined up to a unitary element of $\mathcal{A}$: if $\mathbf{x}_0'$ is another such generator, then there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}_0' = \mathbf{x}_0 u$. Having fixed a generator for ℓ, as we saw above, $(T\mathcal{AP}_1^\rho)_\ell$ identifies with $\ell^{\perp_\rho}$, and to the tangent vector V corresponds an element $\mathbf{v} \in \ell^{\perp_\rho}$. We define

$$|V|_\ell := \|\theta_\rho(\mathbf{v}, \mathbf{v})\|^{1/2}. \tag{25}$$

Note that $|\ |_\ell$ does not depend on the choice of the generator. If we choose $\mathbf{x}_0' = \mathbf{x}_0 u$ instead, the tangent vector V is represented by $\mathbf{v}' = \mathbf{v}u \in \ell^{\perp_\rho}$, and therefore

$$\|\theta_\rho(\mathbf{v}', \mathbf{v}')\| = \|\theta_\rho(\mathbf{v}u, \mathbf{v}u)\| = \|u^*\theta_\rho(\mathbf{v}, \mathbf{v})u\| = \|\theta_\rho(\mathbf{v}, \mathbf{v})\|.$$

Next, recall from Lemma that θ_ρ is negative definite (non degenerate), and therefore the expression (25) above defines a proper norm in $(T\mathcal{AP}_1^\rho)_\ell$.

Remark 11 If $V \in (T\mathcal{AP}_1^\rho)_{[\mathbf{x}_0]}$, V is represented by some $\mathbf{v} \in [\mathbf{x}_0]^{\perp_\rho}$; since $\mathbf{y}_0 = \begin{pmatrix} (x_1^*)^{-1}x_2^* \\ 1 \end{pmatrix}$, for $\mathbf{x}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, is a generator of $[\mathbf{x}_0]^{\perp_\rho}$, there exists $a \in \mathcal{A}$ such that $\mathbf{v} = \mathbf{y}_0 a$. Then

$$|V|_{[\mathbf{x}_0]} = \|\theta_\rho(\mathbf{y}_0 a, \mathbf{y}_0 a)\|^{1/2} = \|a^*(1 - x_2x_1^{-1}(x_2x_1^{-1})^*a\|^{1/2} = \|(1 - |(x_2x_1^{-1})^*|^2)a\|.$$

Note also that the norm of $\mathbf{v} = \mathbf{y}_0 a$ in $\mathcal{A}^2$ is

$$\|\mathbf{y}_0 a\| = \|\langle \mathbf{y}_0 a, \mathbf{y}_0 a\rangle\|^{1/2} = \|(1 + |(x_2 x_1^{-1})^*|^2)a\|.$$

Proposition 6 *For any $\ell \in \mathcal{AP}_1^\rho$, the norm $|\ \ |_\ell$ of $(T\mathcal{AP}_1^\rho)_\ell$ is complete.*

Proof With the current notations, if $V = \mathbf{y}_0 a \in (T\mathcal{AP}_1^\rho)_{[\mathbf{x}_0]}$,

$$|V|_{[\mathbf{x}_0]} = \|a^*(1 - |(x_2 x_1^{-1})^*|^2)a\|^{1/2}.$$

Note that since $1 - |(x_2 x_1^{-1})^*|^2 \le 1 + |(x_2 x_1^{-1})^*|^2$, one has that

$$a^*(1 - |(x_2 x_1^{-1})^*|^2)a \le a^*(1 + |(x_2 x_1^{-1})^*|^2)a,$$

and therefore,

$$|V|_{[\mathbf{x}_0]} = \|a^*(1 - |(x_2 x_1^{-1})^*|^2)a\|^{1/2} \le \|a^*(1 + |(x_2 x_1^{-1})^*|^2)a\|^{1/2} = \|\mathbf{y}_0 a\|.$$

On the other hand

$$\begin{aligned}\|\mathbf{y}_0 a\| &= \|(1 + |(x_2 x_1^{-1})^*|^2)^{1/2} a\| \\ &= \|(1 + |(x_2 x_1^{-1})^*|^2)^{1/2}(1 - |(x_2 x_1^{-1})^*|^2)^{-1/2}(1 - |(x_2 x_1^{-1})^*|^2)^{1/2} a\| \\ &\le \|(1 + |(x_2 x_1^{-1})^*|^2)^{1/2}(1 - |(x_2 x_1^{-1})^*|^2)^{-1/2}\|\ |V|_{[\mathbf{x}_0]}.\end{aligned}$$

It follows that on $(T\mathcal{AP}_1^\rho)_{\ell]} \simeq [\mathbf{y}_0]$, the metric $|\ \ |_\ell$ and the norm of $\mathcal{A}^2$ are equivalent. Since $[\mathbf{y}_0]$ is closed in $\mathcal{A}^2$, it is complete, and the proof follows. □

The distribution $\mathcal{AP}_1^\rho \ni \ell \mapsto |\ \ |_\ell$ is clearly continuous. Thus, $\mathcal{AP}_1^\rho$ is endowed with a Finsler metric.

The following result is tautological, but of the utmost importance for our discussion:

Theorem 9 *The Finsler metric defined in (25) is invariant under the action of $\mathcal{U}_\rho$.*

Proof Pick $\ell = [\mathbf{x}_0] \in \mathcal{AP}_1^\rho$, $V \in (T\mathcal{AP}_1^\rho)_\ell$ (as before, V is identified to $\sim \mathbf{y}_0 a$) and $g \in \mathcal{U}_\rho$. The action of $\mathcal{U}_\rho$ on $\mathcal{AP}_1^\rho$ induces an action on the tangent spaces. Since g preserves θ_ρ,

$$g([\mathbf{y}_0]^{\perp_\rho}) = [g\mathbf{y}_0]^{\perp_\rho}.$$

Then

$$|gV|_{g[\mathbf{x}_0]} = \|\theta_\rho(g(\mathbf{y}_0 a), g(\mathbf{y}_0 a))\|^{1/2} = \|\theta_\rho(\mathbf{y}_0 a, \mathbf{y}_0 a)\|^{1/2} = |V|_{[\mathbf{x}_0]}.$$

□

3.2.1 Invariant Metric in $\mathcal{AP}_1^\rho$

Lemma 3 *The differential of the map $\mathcal{AP}_1^\rho \to \mathcal{D}$, $[\mathbf{x}] \mapsto x_2 x_1^{-1}$, at $[\mathbf{e_1}]$ is the tangent map*

$$\left[\begin{pmatrix} 0 \\ x \end{pmatrix}\right] \longmapsto x\ , \quad x \in \mathcal{A}.$$

Proof Fix $x \in \mathcal{A}$. We use the commutative diagram (28) in Chap. 3. Let $\mathbf{x}(t) \in \mathcal{K}_\rho$ be a smooth curve such that $\mathbf{x}(0) = \mathbf{e_1}$, and the derivative of $[\mathbf{x}(t)]$ (in $\mathcal{AP}_1^\rho$) at $t = 0$ is $\left[\begin{pmatrix} 0 \\ x \end{pmatrix}\right]$. Then $x_1(0) = 1$, $x_2(0) = 0$, $\dot{x}_2(0) = x$. If we map $\mathbf{x}(t)$ onto $\mathcal{D}$, and differentiate at $t = 0$ we get:

$$\frac{d}{dt} x_2(t) x_1^{-1}(t)|_{t=0} = \dot{x}_2(0) x_1^{-1}(0) - x_2(0) x_1^{-1}(0) \dot{x}_1(0) x_1^{-1}(0) = x,$$

which proves our claim. □

Theorem 10 *The identification $[\mathbf{x}] \longleftrightarrow x_2 x_1^{-1}$ between $\mathcal{AP}_1^\rho$ and $\mathcal{D}$, is isometric.*

Proof The group $\mathcal{U}_\rho$ acts isometrically on both $\mathcal{AP}_1^\rho$ and $\mathcal{D}$. Therefore, it suffices to show that the differential at $[\mathbf{e_1}]$ is an isometry. By Lemma 3, this map is

$$\left[\begin{pmatrix} 0 \\ x \end{pmatrix}\right] \longmapsto x.$$

The norm of $\left[\begin{pmatrix} 0 \\ x \end{pmatrix}\right]$ is computed via the identification $(T\mathcal{AP}_1^\rho)_{[\mathbf{e_1}]} \simeq [\mathbf{e_1}]^{\perp_\rho} = [\mathbf{e_2}]$, which sends $\left[\begin{pmatrix} 0 \\ x \end{pmatrix}\right]$ to $\mathbf{e_2}x$. The norm (in $\mathcal{A}^2$) of this element is

$$\|x^* \theta_\rho(\mathbf{e_2}, \mathbf{e_2}) x\|^{1/2} = \|x^* x\|^{1/2} = \|x\|.$$

On the other hand, the norm of x as a tangent element of $\mathcal{D}$ at 0 is the usual norm $\|x\|$ (in $\mathcal{A}$). □

3.3 *Limit Points of Geodesics*

One of our concerns in computing the geodesic δ, and the above results on the polar decomposition, is to establish the following result:

Theorem 11 *For $z \in \mathcal{D}$, let δ be the unique geodesic of $\mathcal{D}$ such that $\delta(0) = 0$ and $\delta(1) = z$. Put $z = \omega|z|$ the polar decomposition (i.e., $\omega \in \mathcal{A}^{**}$); then*

$$\mathrm{SOT} - \lim_{t \to \infty} \delta(t) = \omega \quad \textit{and} \quad \mathrm{SOT} - \lim_{t \to -\infty} \delta(t) = -\omega.$$

Proof By formula (31) in Chap. 3, we only need to compute the limit of $\tanh(t|z|)$ when $t \to \pm\infty$. The spectrum $\sigma(|z|)$ is contained in $[0, 1)$. Clearly, for any $s \in [0, 1)$,

$$\lim_{t\to\infty} \tanh(ts) = \begin{cases} 1 & \text{if } s \in (0, 1) \\ 0 & \text{if } s = 0. \end{cases}$$

By Lebesgue's bounded convergence theorem, and the Borel functional calculus for bounded selfadjoint operators, we have that

$$\lim_{t\to\infty} \tanh(t|\alpha|) = \chi_{(0,1)}(|\alpha|) = P_{N(\alpha)^\perp}.$$

Then,

$$\lim_{t\to\infty} \delta(t) = \omega P_{N(\alpha)^\perp} = \omega.$$

Similarly, using that $\lim_{t\to-\infty} \tanh(st) = \begin{cases} -1 & \text{if } s \in (0, 1) \\ 0 & \text{if } s = 0 \end{cases}$, we get that

$$\lim_{t\to-\infty} \delta(t) = -\omega P_{N(\alpha)^\perp} = -\omega.$$

□

This geometric role of the partial isometry ω in the polar decomposition of $z \in \mathcal{D}$ (or, more generally, of every $z \in \mathcal{A} \setminus \{0\}$) has not been noticed before, to the authors' knowledge.

In order to compute the limit points of arbitrary geodesics, it will be useful to extend the action of $\mathcal{U}_\rho$ to the strong operator border

$$\partial\mathcal{D} := \{a \in \mathcal{A}^{**} : \|a\| = 1\}$$

of $\mathcal{D}$, i.e., to define $g \cdot a$ for $a \in \mathcal{A}^{**}$ with $\|a\| = 1$.

Lemma 4 *If $g \in \mathcal{U}_\rho$ and $a \in \partial\mathcal{D}$, then $g_{11} + g_{12}a$ is invertible in $\mathcal{A}^{**}$.*

Proof Note that

$$g_{11} + g_{12}a = \langle \mathbf{e_1}, g \begin{pmatrix} 1 \\ a \end{pmatrix} \rangle.$$

Using the polar decomposition $g = u|g|$, $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$,

$$\langle \mathbf{e_1}, g \begin{pmatrix} 1 \\ a \end{pmatrix} \rangle = u_1^* \langle \mathbf{e_1}, |g| \begin{pmatrix} 1 \\ a \end{pmatrix} \rangle,$$

i.e., we may suppose $g \geq 0$, $g = \begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix}$, for $b \in \mathcal{A}$. Then

$$g_{11} + g_{12}a = (1+b^*b)^{1/2} + b^*a = (1+b^*b)^{1/2}(1 + (1+b^*b)^{-1/2}b^*a).$$

It suffices to show that $\|(1+b^*b)^{-1/2}b^*a\| < 1$. Note that, since $\|a\| = 1$,

$$\|(1+b^*b)^{-1/2}b^*a\|^2 \leq \|(1+b^*b)^{-1/2}b^*\|^2 = \|(1+b^*b)^{-1/2}b^*b(1+b^*b)^{-1/2}\|$$

$$= \max\{f(t) : t \in \sigma(b^*b)\},$$

for $f(t) = \frac{t}{(1+t)}$. Clearly, this number is strictly less than 1. □

Proposition 7 *If $a \in \partial\mathcal{D}$, and $g \in \mathcal{U}_\rho$, then*

$$g \cdot a := (g_{21} + g_{22}a)(g_{11} + g_{12}a)^{-1} \in \partial\mathcal{D},$$

defines a left action of $\mathcal{U}_\rho$ on $\partial\mathcal{D}$.

Proof A density argument (or a proof similar as in the previous lemma), shows that if $x \in \mathcal{A}^{**}$ with $\|x\| < 1$, then $g \cdot x$, defined as above, also satisfies $\|g \cdot x\| < 1$, and defines an action on the unit ball of $\mathcal{A}^{**}$. Let $a \in \partial\mathcal{D}$. Then, by Kaplansky's density theorem, there exists a sequence $a_n \in \mathcal{D}$ such that $a_n \to a$ in the strong operator topology. We claim that

Lemma 5 *$g \cdot a_n \to g \cdot a$ strongly.*

Proof Consider the polar decomposition $g = u|g|$. We check first that $|g| \cdot a_n \to |g| \cdot a$ strongly. As before, $g = \begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix}$. Then

$$|g| \cdot a_n = (b + (1+bb^*)^{1/2}a_n)((1+b^*b)^{1/2} + b^*a_n)^{-1}.$$

Clearly, $b + (1+bb^*)^{1/2}a_n \to b + (1+bb^*)^{1/2}a$ and $(1+b^*b)^{1/2} + b^*a_n \to (1+b^*b)^{1/2} + b^*a$ strongly. Moreover,

$$((1+b^*b)^{1/2} + b^*a_n)^{-1} = (1 + (1+b^*b)^{-1/2}b^*a_n)^{-1}(1+b^*b)^{-1/2}.$$

Let us show that the norms of these inverses are uniformly bounded. It suffices to see that the norms $\|(1 + (1+b^*b)^{-1/2}b^*a_n)^{-1}\|$ are uniformly bounded. Denote $d_n = (1+b^*b)^{-1/2}b^*a_n$. Then, as seen above,

$$\|d_n\|^2 \leq \max\{f(t) : t \in \sigma(b^*b)\} = r^2 < 1.$$

Thus,

$$\|(1 + (1+b^*b)^{-1/2}b^*a_n)^{-1}\| \leq \frac{1}{1-r}.$$

Therefore the inverses $(1+(1+b^*b)^{-1/2}b^*a_n)^{-1}$ converge strongly to $(1+(1+b^*b)^{-1/2}b^*a)^{-1}$. Then, clearly, $c_n := |g|\cdot a_n$ converge strongly to $c := |g|\cdot a$. Since $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, it is clear that

$$g\cdot a_n = u\cdot(|g|\cdot a_n) = u_2 c_n u_1^* \to u_2 c u_1^* = g\cdot a.$$

□

Let us proceed with the proof of Proposition 7. Since $\|a_n\| < 1$, we know that $\|g\cdot a_n\| < 1$. From Lemma 5, it follows that $\|g\cdot a\| \le 1$. Suppose that $\|g\cdot a\| < 1$. Using again the fact that $\mathcal{U}_\rho$ acts on $\mathcal{D}$, this would imply that

$$g^{-1}\cdot(g\cdot a) = a \in \mathcal{D},$$

a contradiction. Thus, $\|g\cdot a\| = 1$.

The fact that this rule defines, indeed, a left action, follows from similar density arguments. □

Using this result, we can compute the limit points of arbitrary geodesics. Since the action of $\mathcal{U}_\rho$ is transitive, given $z_1, z_2 \in \mathcal{D}$, there exists $g \in \mathcal{U}_\rho$ such that $g\cdot z_1 = 0$.

Corollary 1 *Let $z_0, z_1 \in \mathcal{D}$ and $g \in \mathcal{U}_\rho$ such that $g\cdot z_0 = 0$. Let δ be the unique geodesic of $\mathcal{D}$ such that $\delta(0) = z_0$ and $\delta(1) = z_1$. Denote by $\dot\delta_0$ the initial velocity of δ. Then*

$$\mathrm{SOT}-\lim_{t\to+\infty}\delta(t) = g\cdot\omega_0 \text{ and } \mathrm{SOT}-\lim_{t\to-\infty}\delta(t) = g\cdot(-\omega_0),$$

where $\omega_0 \in \mathcal{A}^{**}$ is the partial isometry in the polar decomposition of $\dot\delta_0$: $\dot\delta_0 = \omega_0|\dot\delta_0|$.

Remark 12 In order to identify these limit points in $\mathcal{D}$, following the notation of the above corollary, note that if $g = |g^*|v = \begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix}\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ (the reversed polar decomposition), then $v\cdot\omega_0 = v_2\omega_0 v_1^*$ is a partial isometry. Therefore, the limit points of geodesics are elements in $\partial\mathcal{D}$ of the form

$$(b+(1+bb^*)^{1/2}\omega)((1+b^*b)^{1/2}+b^*\omega)^{-1} \text{ and } (b-(1+bb^*)^{1/2}\omega)((1+b^*b)^{1/2}-b^*\omega)^{-1},$$

where $b \in \mathcal{A}$ is arbitrary and $\omega \in \mathcal{A}^{**}$ is a partial isometry.

Note that not every partial isometry in $\mathcal{A}^{**}$ occurs in the polar decomposition of an element in $\mathcal{D}$. For instance, if $\mathcal{A} = C([0,1])$ (continuous functions in the unit interval), the polar decomposition of $f \in \mathcal{A}$ is $f = w|f|$, where $w \in L^\infty(0,1)$ is given by $w(t) = \begin{cases} f(t)/|f(t)| \text{ if } f(t) \neq 0 \\ 0 \text{ if } f(t) = 0 \end{cases}$. An arbitrary partial isometry in $L^\infty(0,1)$ is a measurable function whose values are zero or complex numbers of modulus 1. The set of zeros of such a function is an arbitrary measurable set, whereas there are no proper (non unitary) partial isometries in $C([0,1])$.

Another way to study the limit points of geodesics, is by using the Borel subgroup $\mathcal{B} \subset \mathcal{U}_\rho$. This is the conjugate group of $\mathcal{B}' \subset \mathcal{U}_{\rho'}$ defined before (Definition 5 in Chap. 3), by means of the unitary matrix U. Indeed, since the action of this group is transitive in $\mathcal{D}$, any limit point of a geodesic is either of the form $g \cdot v$ or $g \cdot (-v)$, for $g \in \mathcal{B}$. Consider the following example:

Example 12 Suppose that $\mathcal{A}$ is a von Neumann algebra, and let $p \neq 0$ be a projection in $\mathcal{A}$. Write $b = \begin{pmatrix} \frac{g+\hat{g}}{2} - \hat{g}x & \frac{g-\hat{g}}{2} - \hat{g}x \\ \frac{g-\hat{g}}{2} + \hat{g}x & \frac{g+\hat{g}}{2} + \hat{g}x \end{pmatrix}$, which is the general form of an element in $\mathcal{B}$ (here we put $\hat{g} = (g^*)^{-1}$). Let us compute $b \cdot p$. After straightforward computations,

$$b \cdot p = \left(g(1+p) + \hat{g}\,(-1+p+2x(1+p))\right)\left(g(1+p) + \hat{g}\,(1-p-2x(1+p))\right)^{-1}.$$

Note that $1+p$ is invertible and that $(1-p)(1+p)^{-1} = 1-p$. Then

$$b \cdot p = \left(1 + \hat{g}(p-1+2x)g^{-1}\right)\left(1 + \hat{g}(1-p-2x)g^{-1}\right)^{-1}.$$

Denote $\alpha = \hat{g}(p-1)g^{-1}$ and $\beta = 2\hat{g}xg^{-1}$. Observe that α is a non-invertible selfadjoint element, $\alpha \leq 0$ and its range is proper and closed; β is an arbitrary anti-Hermitien element. Then

$$b \cdot p = (1+\alpha+\beta)\,(1-(\alpha+\beta))^{-1}.$$

Note that $1-(\alpha+\beta)$ is invertible because $Re(1-(\alpha+\beta)) = 1-\alpha \geq 1$. If one picks $p = 1$, then $\alpha = 0$ and

$$b \cdot 1 = (1+\beta)(1-\beta)^{-1},$$

which is a unitary operator such that -1 does not belong to its spectrum. In particular, this shows that the action of $\mathcal{B}$ ceases to be transitive in $\partial\mathcal{D}$.

Our next result shows a necessary condition for an element $a \in \mathcal{A}^{**}$ with $\|a\| = 1$ to be a limit point of a geodesic of $\mathcal{D}$.

Proposition 8 *If $a \in \mathcal{A}^{**}$ is the limit point at $+\infty$ of a geodesic in $\mathcal{D}$, then $1 - a^*a = hqh$, where $h \in G^+$ and $q \in \mathcal{A}^{**}$ is a projection. In particular, not every element of norm 1 in $\mathcal{A}^{**}$ is the limit point of a geodesic: such elements satisfy that the defect element $1 - a^*a$ has closed range.*

Proof The element $a \in \mathcal{A}^{**}$ is the limit point of a geodesic if and only if there exists $g \in \mathcal{U}_\rho$ and a partial isometry $\omega \in \mathcal{A}^{**}$ such that $a = g \cdot \omega$. The definition of the action implies that this equality can be read as an usual matrix equality

$$g\begin{pmatrix} 1 \\ \omega \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

with $b_1 \in G$ and $b_2 b_1^{-1} = a$. Using the form θ_ρ, and the fact that $g \in \mathcal{U}_\rho$,

$$\theta_\rho(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}) = \theta_\rho(\begin{pmatrix} 1 \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \end{pmatrix}),$$

i.e.

$$b_1^* b_1 - b_2^* b_2 = 1 - \omega^* \omega = q',$$

which is a projection in $\mathcal{A}^{**}$. Let $b_1 = u|b_1|$ be the polar decomposition, with u unitary. Then $au = b_2 b_1^{-1} u = b_2 |b_1|^{-1}$. Thus,

$$q' = b_1^* b_1 - b_2^* b_2 = |b_1|(1 - |b_1|^{-1} b_2^* b_2 |b_1|^{-1})|b_1| = |b_1|(1 - u^* a^* a u)|b_1|,$$

i.e.,

$$1 - a^* a = u|b_1|^{-1} q' |b_1|^{-1} u^* = hqh,$$

for $q = uq'u^*$ a projection in $\mathcal{A}^{**}$ and $h = u|b_1|^{-1}u^* \in G^+$. □

Remark 13 The characterization of the partial isometries which appear in the polar decompositions of all limit points $a \in \mathcal{A}^{**}$ is an interesting open problem.

3.4 *Operator Cross Ratio in the Hyperbolic Part of the Projective Line*

Here we state the main result of this part of the book, relating the metric of $\mathcal{A}\mathbb{P}_1^\rho$ introduced in Sect. 3.2, with the so called operator cross ratio, as defined in the Grassmann manifold of a Hilbert space by Zelikin [16]. We shall apply these ideas to the rank one submodules in $\mathcal{A}\mathbb{P}_1^\rho$. To this effect, the isometry between $\mathcal{A}\mathbb{P}_1^\rho$ and the disk $\mathcal{D}$ will be important.

Consider $\ell = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right] \in \mathcal{A}\mathbb{P}_1^\rho$, for $z \in \mathcal{D}$. Let $z = \omega|z|$ be the polar decomposition. Let δ be the geodesic of $\mathcal{A}\mathbb{P}_1^\rho$ such that $\delta(0) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ and $\delta(1) = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right]$. Equivalently, regarded in $\mathcal{D}$: $\delta(0) = 0$ and $\delta(1) = z$. As seen in Sect. 3.3,

$$\text{SOT} - \lim_{t \to +\infty} \delta(t) = \omega \text{ and SOT} - \lim_{t \to -\infty} \delta(t) = -\omega.$$

Four points are determined: $-\omega$, 0, z, ω, or better, four rank one submodules

$$\ell_{-\infty} := \left[\begin{pmatrix} 1 \\ -\omega \end{pmatrix}\right], \ell_0 := \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \ell = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right], \ell_{+\infty} := \left[\begin{pmatrix} 1 \\ \omega \end{pmatrix}\right],$$

where the limit lines lie in $\partial \mathcal{A}\mathbb{P}_1^\rho$.

In Sect. 3.1 we defined the operator cross ratio of four elements in $\mathcal{A}\mathbb{P}_1$, as a (possibly empty) set of module endomorphisms, following ideas of Zelikin [16]. Here we compute the operator cross ratio $CR(\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty})$, proving that it is non empty, and that there exists a natural ℓ-endomorphism to choose from this set.

Recall that elements of $CR(\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty})$ are (module) endomorphisms of ℓ, defined as the composition of *the projection from ℓ to ℓ_0 parallel to $\ell_{-\infty}$, followed by the projection from ℓ_0 to ℓ parallel to $\ell_{+\infty}$*.

In coordinates, by choosing generators in the respective submodules

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda, \ \begin{pmatrix} 1-\lambda \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} \mu.$$

Then $1 - \lambda = \mu$ and $z = -\omega\mu$. Then $\omega|z| = -\omega\mu$. If z is invertible (and then ω is unitary) this implies $\mu = -|z|$, otherwise this is just one possible solution. Non uniqueness of solutions of these equations reflect the geometric fact that the modules ℓ_0 and ℓ_∞ may not be in direct sum. Explicitly, all solutions of these equations are of the form

$$\lambda = 1 + |z| - \Omega \ , \quad \mu = -|z| + \Omega,$$

where $\Omega \in \mathcal{A}^{**}$ is such that $\omega\Omega = 0$. In particular, $|z|\Omega = |z|\omega^*\omega\Omega = 0$. We choose the solution with $\Omega = 0$. Note that $\lambda = 1 + |z|$, and therefore the first projection in the above composition is given by

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1+|z| \\ 0 \end{pmatrix}.$$

Next

$$\begin{pmatrix} 1+|z| \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ z \end{pmatrix} \gamma \ , \ \begin{pmatrix} 1+|z|-\gamma \\ -z\gamma \end{pmatrix} = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \epsilon.$$

So that $1 + |z| - \gamma = \epsilon$ and $-z\gamma = \omega\epsilon$, and then (the unique solution if ω is unitary, or a possible solution that we choose, otherwise) $1 + |z| - \gamma = -|z|\omega$, i.e.,

$$\gamma = (1 - |z|)^{-1}.$$

Other solutions of the above equation are of the form

$$\gamma = (1 - |z|)^{-1} + (1 - |z|)^{-1}\Omega',$$

where $\Omega' \in \mathcal{A}^{**}$ is such that $\omega\Omega' = 0$. In general, the possible endomorphisms $\ell \to \ell$ are given (in these coordinates) by

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z| + \Omega)(1 - |z|)^{-1}(1 + \Omega') = (1 + |z|)(1 - |z|)^{-1} + \Omega' + \Omega(1 - |z|)^{-1} + \Omega\Omega',$$

where we use that $\omega\Omega = \omega\Omega' = 0$, and thus $(1 \pm |z|)^{\pm 1}\Omega = \Omega$ (and the same for Ω').

As noted, if z is invertible, there is a unique solution with $\Omega = \Omega' = 0$. Our choice of cross ratio, picking $\Omega = \Omega' = 0$ in any case, is justified below.

Remark 14 The following fact is known (see for instance [19]). Let $a_n \in \mathcal{A}$ with $\|a_n\| \leq 1$. If $a_n \to a$ strongly, then $|a_n| \to |a|$ strongly.

Remark 15 Dixmier and Marechal [20] proved that the set on invertible elements of a von Neumann algebra is strong operator dense in the algebra. The argument in [20] proceeds as follows. Let $a = u|a|$ be the polar decomposition of a ($u \in \mathcal{A}$). First, the algebra $\mathcal{A}$ is factored in its finite and properly infinite parts. In the finite part u can be chosen unitary. In the properly infinite part, one readily sees that it suffices to consider the cases in which u is an isometry or a co-isometry. In the case that u is an isometry, Dixmier and Maréchal prove that u is the strong limit of unitaries u_n. If u is a co-isometry, they show that there exist invertible elements g_n which converge strongly to u, with norms $\|g_n\| = 1$ (this is clear in the proof, though it is not stated in their result). Summarizing, if $\mathcal{A}$ is a von Neumann algebra, and $a \in \mathcal{A}$, there exist $g_n \in G$ with $\|g_n\| \leq \|a\|$ such that

$$\text{SOT} - \lim_{n\to\infty} g_n = a.$$

Using this fact, it is clear that, if $\mathcal{A}$ is a von Neumann algebra, and $z \in \mathcal{D}$, then there exist $z_n \in G$ with $\|z_n\| \leq \|z\|$, such that $z_n \to z$ strongly.

Proposition 9 *Let $z_n, z \in \mathcal{D}$ such that $z_n \to z$ strongly and $\|z_n\| \leq \|z\|$. Then*

$$(1 + |z_n|)(1 - |z_n|)^{-1} \to (1 + |z|)(1 - |z|)^{-1} \textit{ strongly}.$$

Proof First note that $|z_n| \to |z|$ strongly. Then $1 \pm |z_n|$ converges strongly to $1 \pm |z|$. Clearly $\|1 + |z_n|\| \leq 2$. Let us check that also $\|(1 - |z_n|)^{-1}\|$ are uniformly bounded:

$$\|(1 - |z_n|)^{-1}\| = \|\sum_{k=0}^{\infty} |z_n|^k\| \leq \sum_{k=0}^{\infty} \|z_n^k\| \leq \frac{1}{1 - \|z\|} < \infty.$$

Therefore, $(1 - |z_n|)^{-1} \to (1 - |z|)^{-1}$ strongly, and since the product is strongly continuous on norm bounded sets, the proof follows. □

Definition 9 Let $z \in \mathcal{D}$. We define $cr(0,z) \in CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_\infty)$, for $\ell_z = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right]$, as the endomorphism

$$cr(0,z) : \ell_z \to \ell_z\ , \quad cr(0,z)\Big(\begin{pmatrix} 1 \\ z \end{pmatrix} a\Big) = \begin{pmatrix} 1 \\ z \end{pmatrix} (1+|z|)(1-|z|)^{-1} a,$$

for $a \in \mathcal{A}$.

We use the action of $\mathcal{U}_\rho$ to extend this definition to any pair $z_0 \neq z_1 \in \mathcal{D}$.

Definition 10 Let $z_0, z_1 \in \mathcal{D}$, $z_0 \neq z_1$. Pick $g \in \mathcal{U}_\rho$ such that $g \cdot 0 = z_0$ and denote $z = g^{-1} \cdot z_1$. We define

$$cr(z_0, z_1) = g\ cr(0,z) g^{-1}.$$

Before checking that the definition does not depend on the choice of g, we remark the following. Let δ be the unique geodesic of $\mathcal{D}$ such that $\delta(0) = z_0$ and $\delta(1) = z_1$, and let

$$z_{-\infty} = \text{SOT} - \lim_{t \to -\infty} \delta(t) \ \text{ and } z_{+\infty} = \text{SOT} - \lim_{t \to +\infty} \delta(t).$$

Then $cr(z_0, z_1) \in CR(\ell_{z_{-\infty}}, \ell_{z_0}, \ell_{z_1}, \ell_{z_{+\infty}})$, because g is a module homomorphism which maps ℓ_z onto ℓ_{z_1}. Indeed, if $\mathbf{x} = \begin{pmatrix} 1 \\ z \end{pmatrix} a \in \ell_z$, then clearly

$$g\mathbf{x} = \begin{pmatrix} 1 \\ g \cdot z \end{pmatrix} (g_{11} + g_{12} z) a = \begin{pmatrix} 1 \\ z_1 \end{pmatrix} (g_{11} + g_{12} z) a \in \ell_{z_1}.$$

Let us check that $cr(z_1, z_2)$ is well defined, i.e., that it does not depend on the choice of g.

To prove this, recall that if $k \in \mathcal{U}_\rho$ satisfies $k \cdot 0 = 0$, then $k = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, with $u_1, u_2 \in \mathcal{U}_\mathcal{A}$.

Proposition 10 *With the above notations, the endomorphism*

$$cr(z_0, z_1) \in CR(\ell_{z_{-\infty}}, \ell_{z_0}, \ell_{z_1}, \ell_{z_{+\infty}})$$

does not depend on the choice of g. Namely, if $h \in \mathcal{U}_\rho$ satisfies $h \cdot 0 = z_0$, and $z' = h^{-1} \cdot z_1$, then

$$h\ cr(0,z') h^{-1} = g\ cr(0,z) g^{-1}.$$

Proof Since $h \cdot 0 = g \cdot 0$, it follows that $(g^{-1}h) \cdot 0 = 0$, and therefore $h = g \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, for $u_1, u_2 \in \mathcal{U}_{\mathcal{A}}$. Then

$$z' = h^{-1} \cdot z_1 = \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \cdot (g^{-1} \cdot z_1) = \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \cdot z = u_2^* z u_1,$$

and

$$h\, cr(0, z')h^{-1} = g \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} cr(0, u_2^* z u_1) \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} g^{-1}.$$

Thus, we must show that $\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} cr(0, u_2^* z u_1) \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} = cr(0, z)$. Let us see how the left-hand side endomorphism transforms the element $\begin{pmatrix} 1 \\ z \end{pmatrix} a \in \ell_z$. First, it is sent to

$$\begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} a = \begin{pmatrix} 1 \\ u_2^* z u_1 \end{pmatrix} u_1^* a.$$

The map $cr(0, z')$ maps this element to

$$\begin{pmatrix} 1 \\ u_2^* z u_1 \end{pmatrix} (1 + |u_2^* z u_1|)(1 - |u_2^* z u_1|)^{-1} u_1^* a.$$

Note that $|u_2^* z u_1| = ((u_2^* z u_1)^* u_2^* z u_1)^{1/2} = (u_1^* z^* z u_1)^{1/2} = u_1^* |z| u_1$. Therefore, the above element equals

$$\begin{pmatrix} u_1^* \\ u_2^* z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} a.$$

Finally, multiplying on the left by the matrix $\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$ yields

$$\begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} a = cr(0, z) \begin{pmatrix} 1 \\ z \end{pmatrix} a.$$

□

Suppose that $\mathcal{A}$ is a von Neumann algebra. If $z \in \mathcal{D}$ is non invertible, there exist $z_n \in \mathcal{D}$ which are invertible such that $z_n \to z$ strongly and $\|z_n\| \leq \|z\|$. Then $cr(0, z_n)$, $cr(0, z)$ are endomorphisms of different submodules. In order to compare them, we can regard them as $\mathcal{A}$-module morphisms of $\mathcal{A}^2$, embedding each module in $\mathcal{A}^2$ using the θ_ρ-orthogonal projections $p_{\ell_{z_n}}$, p_{ℓ_z} onto the submodules ℓ_{z_n}, ℓ_z, respectively. For $z' \in \mathcal{D}$,

$$p_{\ell_{z'}}(\mathbf{x}) = (1 - |z'|^2)^{-1/2} \theta_\rho(\begin{pmatrix} 1 \\ z' \end{pmatrix}, \mathbf{x}) \begin{pmatrix} 1 \\ z' \end{pmatrix} (1 - |z'|^2)^{-1/2}.$$

Proposition 11 *Let $\mathcal{A}$ be a von Neumann algebra.*

1. *If $z \in \mathcal{D} \cap G_{\mathcal{A}}$, the set $CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_\infty)$ consists of a single element $cr(0, z)$.*
2. *If $z \in \mathcal{D}$ is non invertible, there exist $z_n \in \mathcal{D}$ which are invertible such that $z_n \to z$ strongly and $\|z_n\| \leq \|z\|$. Then, with the above notations,*

$$cr(0, z_n)p_{\ell_{z_n}}(\mathbf{x}) \to cr(0, z)p_{\ell_z}(\mathbf{x})$$

strongly in $\mathcal{A}^2$.

Proof By Remark 9, we know that $(1 + |z_n|)(1 - |z_n|)^{-1} \to (1 + |z|)(1 - |z|)^{-1}$ strongly. By a similar argument, it also holds that $(1 - |z_n|^2)^{-1/2} \to (1 - |z|^2)^{-1/2}$ strongly. Also these operators are uniformly bounded. Therefore, using that the product is strongly continuous in bounded sets, our claim follows. □

Observe that in the von Neumann algebra case $cr(0, z)$ is then uniquely defined when z is invertible, and is defined by strong continuity in the remaining cases.

Remark 16 As a corollary we get that, even if the set $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ may be empty for general $\ell_1, \ell_2, \ell_3, \ell_4$, the particular set $CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_\infty)$ is not, and $cr(0, z)$ is a distinguished element of this set.

As a first approximation of the deep relationship between the cross ratio and the metric in $\mathcal{A}\mathbb{P}^\rho_1$, we can state the following result. It can be regarded as the scalar version of our main result, here we state the equality between the norm of the endomorphism $cr(0, z)$ and the distance between 0 and z in the Poincaré disk $\mathcal{D}$.

Theorem 13 *Let $z \in \mathcal{D}$, then*

$$\frac{1}{2}\|cr(0, z)\|_{\mathcal{B}(\ell_z)} = d(0, z),$$

where $\| \ \|_{\mathcal{B}(\ell_z)}$ denotes the norm of operators acting in $\ell_z \subset \mathcal{A}^2$.

Proof Choose for ℓ_z the unital basis $\mathbf{e_z} = \begin{pmatrix} (1 - z^*z)^{-1/2} \\ z(1 - z^*z)^{-1/2} \end{pmatrix}$. Then for any $\mathbf{x} = \mathbf{e_z}a \in \ell_z$,

$$cr(0, z)\mathbf{x} = \mathbf{e_z} \log((1 + |z|)(1 - |z|)^{-1}a,$$

and thus

$$\langle cr(0, z)\mathbf{x}, cr(0, z)\mathbf{x}\rangle = a^* \log((1 + |z|)(1 - |z|)^{-1}\langle \mathbf{e_z}, \mathbf{e_z}\rangle \log((1 + |z|)(1 - |z|)^{-1}a$$

$$= a^*(\log((1 + |z|)(1 - |z|)^{-1})^2 a.$$

Since $(\log((1 + |z|)(1 - |z|)^{-1})^2 \leq \| \log((1 + |z|)(1 - |z|)^{-1}\|^2$, it follows that

$$a^*(\log((1 + |z|)(1 - |z|)^{-1})^2 a \leq a^*a\| \log((1 + |z|)(1 - |z|)^{-1}\|^2,$$

and therefore

$$\|\langle cr(0,z)\mathbf{x}, cr(0,z)\mathbf{x}\rangle\|^{1/2} \leq \|\log((1+|z|)(1-|z|)^{-1}\|\|a^*a\|^{1/2}$$

$$= \|\log((1+|z|)(1-|z|)^{-1}\|\|\mathbf{x}\|.$$

This implies that $\|cr(0,z)\|_{\mathcal{B}(\ell_z)} \leq \|\log((1+|z|)(1-|z|)^{-1}\|$. Note that

$$cr(0,z)\mathbf{e_z} = \mathbf{e_z}\log((1+|z|)(1-|z|)^{-1},$$

so that

$$\|cr(0,z)\mathbf{e_z}\|^2 = \|\langle \mathbf{e_z}\log((1+|z|)(1-|z|)^{-1}, \mathbf{e_z}\log((1+|z|)(1-|z|)^{-1}\|$$

$$= \|\log((1+|z|)(1-|z|)^{-1}\|^2,$$

i.e.,

$$\|cr(0,z)\|_{\mathcal{B}(\ell_z)} = \|\log((1+|z|)(1-|z|)^{-1}\| = 2d(0,z).$$

□

3.5 The Cross Ratio, the Logarithm and the Exponential

We have defined a Hilbertian C-valued structure in $\mathcal{Q}_\rho \simeq \mathcal{AP}_1^\rho$, or, equivalently, in $\mathcal{D}$. In particular, the product

$$\langle \mathbf{Log}_0(z), \mathbf{Log}_0(z)\rangle_0$$

takes values in the set of endomorphisms of $\ell_0 = \left[\begin{pmatrix}1\\0\end{pmatrix}\right]$, where $\mathbf{Log}_0$ is defined in Corollary 7 of Chap. 3. It is a positive module endomorphism (given by multiplying the generator $\mathbf{e_1}$ by a positive element of a). Thus, it has a unique positive square root $\langle \mathbf{Log}_0(z), \mathbf{Log}_0(z)\rangle_0^{1/2}$, which we denote by $mod_0(\mathbf{Log}_0(z))$ of $\mathbf{Log}_0(z)$. Explicitly, in the generator $\mathbf{e_1}$, $mod_0(\mathbf{Log}_0(z))$ consists in multiplying the generator by $\log\left((1+|z|)(1-|z|)^{-1}\right)$.

On the other hand, we saw that, for $z \in \mathcal{D}$, the endomorphism of ℓ_z denoted by $cr(0,z)$, is given by the same coefficient $\log\left((1+|z|)(1-|z|)^{-1}\right)$, which multiplies the generator $\begin{pmatrix}1\\z\end{pmatrix}$ of ℓ_z.

We shall translate the endomorphism $cr(0,z)$ from ℓ_z to ℓ_0 by means of the parallel transport of $\mathcal{AP}_1^\rho$, along the geodesic δ, with $\delta(0) = \ell_0$ and $\delta(1) = \ell_z$ (i.e., the same former δ, which under the identification $\mathcal{D} \simeq \mathcal{AP}_1^\rho$ joins $\delta(0) = 0$ and $\delta(1) = z$ in $\mathcal{D}$: $\delta(t) = \omega\tanh(t|\alpha|)$).

The parallel transport of elements of $\mathcal{D}$ (or $\mathcal{A}\mathbb{P}_1^\rho$) along the geodesic $\delta(t) = e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \cdot 0$, where α is, as in (30) of Chap. 3

$$\alpha = z \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^* z)^k,$$

is given by the left action of the invertible matrix $e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} : \ell_0 \to \ell_{\delta(t)}$. The endomorphism $cr(0, z)$ of ℓ_z is transported to ℓ_0 as

$$cr(0, z)_0 := e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0, z) e^{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} : \ell_0 \to \ell_0.$$

Our main result (for the origin) is the following:

Theorem 14 *With the current notation, if $z \in \mathcal{D}$ (or $\ell_z \in \mathcal{A}\mathbb{P}_1^\rho$),*

$$e^{mod_0(\mathbf{Log}_0(z))} = cr(0, z)_0 \tag{26}$$

or, equivalently,

$$mod_0(\mathbf{Log}_0(z)) = \log(cr(0, z)_0), \tag{27}$$

where the exponential in the first equality is the usual exponential of $\mathcal{A}$, $\log$ in the second equality is the usual logarithm of G^+, and each endomorphism of ℓ_0 is identified with its coefficient in the basis $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Proof Let us prove the first equality. Since we are comparing endomorphisms of ℓ_0, it suffices to show that they carry the generator $\mathbf{e_1}$ to the same element in $\mathcal{A}^2$. Note that

$$cr(0, z)_0(\mathbf{e_1}) = e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0, z) e^{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0, z) \begin{pmatrix} \cosh(|\alpha|) \\ \omega \sinh(|\alpha|) \end{pmatrix};$$

using that $\omega \tanh(|\alpha|) = \delta(1) = z$ (Lemma 3 in Chap. 3), this yields

$$e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0, z) \begin{pmatrix} 1 \\ z \end{pmatrix} \cosh(|\alpha|) = e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} \cosh(|\alpha|).$$

By the same computation that showed that $e^{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix} \cosh(|\alpha|)$ (see the proof of Lemma 3 in Chap. 3), we have that

$$e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cosh(|\alpha|)^{-1},$$

i.e.,

$$cr(0, z)_0(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + |z|)(1 - |z|)^{-1}.$$

On the other hand, the endomorphism $mod_0(\mathbf{Log}_0(z)$ sends $\mathbf{e_1}$ to

$$\mathbf{e_1} \log \left((1 + |z|)(1 - |z|)^{-1}\right),$$

and thus

$$e^{mod_0(\mathbf{Log}_0(z))}(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + |z|)(1 - |z|)^{-1}.$$

□

As in Definition 10, let $z_0 \neq z_1 \in \mathcal{D}$ ($\ell_{z_0} \neq \ell_{z_1} \in \mathcal{A}\mathbb{P}_1^\rho$). Pick $g \in \mathcal{U}_\rho$ such that $g \cdot 0 = z_0$, and denote by $z = g^{-1} \cdot z_1$ as before. Let δ be the geodesic such that $\delta(0) = 0$ and $\delta(1) = z_1$. Then $\delta_{z_0,z_1} = g \cdot \delta$ is the geodesic which joins z_0 and z_1 at $t = 0$ and $t = 1$, respectively. Recall that $cr(z_0, z_1) = g cr(z_0, z_1) g^{-1}$. Likewise, we put

$$\mathbf{Log}_{z_0}(z_1) := g\mathbf{Log}_0(z)g^{-1}, \quad \text{and} \quad mod_{z_0}(z_1) = \langle \mathbf{Log}_{z_0}(z_1), \mathbf{Log}_{z_0}(z_1) \rangle_{z_0}^{1/2},$$

where $\langle \varphi, \psi \rangle_{z_0} = g\langle g\varphi g^{-1}, g\psi g^{-1} \rangle_0 g^{-1}$, and $\mathbf{Log}_{z_0}$ is the inverse of the exponential $exp_{z_0} : (T\mathcal{D})_{z_0} \to \mathcal{D}$. It is not difficult to verify that these definitions do not depend on the choice of g.

Finally, let us denote by $cr(z_0, z_1)_{z_0}$ the parallel transport of $cr(z_0, z_1)$ from ℓ_{z_1} to ℓ_{z_0} along the geodesic δ_{z_0,z_1} (obtained by conjugation as in the case of the origin, by the value at $t = 1$ of the one parameter group in $\mathcal{U}_\rho$ which determines δ_{z_0,z_1}). The $\mathcal{U}_\rho$-covariance of the data involved enables us to prove the following:

Theorem 15 *With the current notations,*

$$mod_{z_0}(\mathbf{Log}_{z_0}(z_1)) = \log \left(cr(z_0, z_1)_{z_0}\right).$$

In particular, $\|\mathbf{Log}_{z_0}(z_1)\|_{z_0} = \|\log(cr(z_0, z_1)_{z_0})\|$.

Remark 17 If we think of $mod_{z_0}(\mathbf{Log}_{z_0}(z_1))$ as an *distance operator* from z_0 to z_1, then the identity in the above theorem shows a projective way of computing this operator distance.

3.6 An Example

Suppose that the algebra $\mathcal{A}$ has a trace **tr** onto a central subalgebra, that is, there exists a C*- subalgebra $\mathcal{B} \subset Z(\mathcal{A})$ of the center of $\mathcal{A}$ and a conditional expectation $\mathbf{tr} : \mathcal{A} \to \mathcal{B}$ satisfying $\mathbf{tr}(xy) = \mathbf{tr}(yx)$ for all $x, y \in \mathcal{A}$. This happens, for instance, if $\mathcal{A}$ is a finite von Neumann algebra.

We can define a Hilbertian $\mathcal{B}$-valued inner product, by means of

$$\langle X, Y\rangle_{\mathbf{tr},q} = -\mathbf{tr}(\theta_\rho(X\mathbf{x}, Y\mathbf{y})).$$

Indeed, since **tr** is tracial, the value of $-\mathbf{tr}(\theta_\rho(X\mathbf{x}, Y\mathbf{y}))$ is independent of the choice of $\mathbf{x} \in \mathcal{K}_\rho$ satisfying $q = p_{\mathbf{x}}$. On the other hand, $cr(z_0, z_1)$ is an element of Γ_{z_0}, which has matrix a in a unital base $\mathbf{x} \in R(q)$, as explained before. We put $cr(z_0, z_1)_{\mathbf{tr}}$, for $\mathbf{tr}(a)$. Clearly, $cr(z_0, z_1)_{\mathbf{tr}}$ does not depend on the basis $\mathbf{x}$. With these notations, the formula in Corollary 15, can be written

$$\langle \mathbf{Log}_{z_0} z_1, \mathbf{Log}_{z_0} z_1\rangle_{\mathbf{tr}}^{1/2} = \log cr(z_0, z_1)_{\mathbf{tr}}, \tag{28}$$

which is an identity involving elements in $\mathcal{B}$.

More specifically, if $\mathcal{A}$ is commutative, we can choose **tr** the identity $\mathcal{A} = \mathcal{B}$, and we have

$$|\mathbf{Log}_{z_0} z_1| = \log cr(z_0, z_1), \tag{29}$$

as elements in $\mathcal{A}$.

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An Introduction to the Non-harmonic Analysis of Pseudo-Differential Operators

Alexander Cardona and Michael Ruzhansky

Abstract These notes are based in the lecture series on the subject given by the second author during the Summer School *Geometric, Algebraic and Topological Methods for Quantum Field Theory*, in Villa de Leyva (Colombia), July 2017. The goal of the lectures was to give a first overview of the algebra of global pseudo-differential operators associated to the boundary value problem on a domain Ω, associated to the spectral data furnished by a differential operator $\mathfrak{L}$ with fixed boundary conditions, giving rise to biorthogonal systems of eigenfunctions in $L^2(\Omega)$.

1 Introduction

Fourier's *théorie de la chaleur* is one of the first examples of application of analysis to physics. It starts with the solution to the differential equation

$$f_t = f_{xx},$$

where $f(t, x)$ is a function on $[0, \infty) \times \mathbb{R}$ with *periodic initial condition* $f(0, x) = f_0(x)$. Classically, the computation and study of Fourier series is known as *harmonic analysis*, and is an extremely useful way to decompose an arbitrary periodic function into a set of simple terms that can be used to approximate solutions to differential equations, to whatever accuracy is desired. Fourier series make use of the orthogonality relationships of the (sine and cosine) functions to decompose a periodic function in a an infinite sum of trigonometric functions. Moreover, any set of functions that

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form a complete orthogonal system give rise to a generalized Fourier series, analogous to the classical one (e.g., using orthogonality of the roots of a Bessel function of the first kind, gives a so-called Fourier–Bessel series).

The heat equation involves a second order partial differential operator, *the Laplace operator* or *Laplacian*, which is ubiquitous in mathematics, and gave rise to important branches of analysis leading to applications in areas as diverse as algebraic topology and probability theory. The study of the analytic properties of the Laplacian on compact Riemannian manifolds (as well as for other fundamental operators, for example its square root, the Dirac operator), and the consequence of its properties on the structure and features of its eigenvalues and its eigenfunctions, gave rise to unexpected applications of harmonic analysis to the study of the geometry and topology of differential manifolds [9] which, in turn, gave a natural language to develop a rather general abstract framework to study the geometry of non-commutative algebras. These developments require algebras of pseudo-differential operators which, as we will see in what follows, come naturally associated to the Laplacian. However, more general frameworks can lead to algebras of pseudo-differential operators defined using the spectral information associated to other (differential or pseudo-differential) operators which, in some cases, may be better suited for particular applications.

These lecture notes are intended to give an introduction to the theory of *nonharmonic pseudo-differential operators*, meaning by that the symbolic calculus and basic algebraic and analytic operations involved in the construction (convolutions, Sobolev space, etc) are adapted to spectral information of a non-necessarily self-adjoint elliptic differential operator. We will survey the main results and applications of [5, 10, 11], where more details, complete references and more general examples can be found. For much more on the methods and applications presented in this review the reader can consult the many open access sources in the website https://ruzhansky.org/.

2 Nonharmonic Fourier Series

The use of Hilbert spaces in mathematics, which can be traced back to the mid nineteenth century, was highly stimulated by its use in the very mathematical foundations of quantum mechanics, as developed by von Neumann [18] in the twentieth century. Hilbert spaces widely used in quantum physics include the (infinite-dimensional) spaces of square-summable sequences of real or complex numbers, denoted by $\ell^2(\mathbb{N})$, or spaces of square-integrable complex-valued functions on $\mathbb{R}$, denoted by $L^2(\mathbb{R})$ in what follows. The usefulness of special basis for such Hilbert spaces was already known when Hilbert, Schmidt and others studied the geometry of those spaces of functions. Many results on general expansions in eigenfunctions (e.g. Parseval relation) where well known at the time from Fourier analysis, and gave rise to important results as the theorems of Riesz and Fischer on the isomorphism of these Hilbert spaces. In this section, we will introduce exponential series of functions called *nonharmonic Fourier series*, which can be seen as perturbations of the usual orthonormal

basis used in harmonic analysis. Nonharmonic Fourier series give us a fist example of more general sets of functions, which can be used to generalize the tools of harmonic analysis to situations in which special symmetries are not assumed; for a complete account of the analysis of nonharmonic Fourier series we refer the reader to [8, 19].

2.1 Biorthogonal Systems and Riesz Basis in Hilbert Spaces

A *biorthogonal system* in a topological vector space V is a pair of sets $\{x_i\}$ and $\{\xi_j\}$, $i, j \in \mathbb{I} \subset \mathbb{Z}$, of elements in V and its dual space V^*, respectively, which satisfy the conditions

$$\xi_j(x_i) = \langle \xi_j, x_i \rangle = 0$$

if $i \neq j$, and $\xi_j(x_i) = \langle \xi_i, x_i \rangle \neq 0$ for all $i \in \mathbb{I}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form coupling V and V^*. For instance, in a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and basis $\{e_n\}$, since $\mathcal{H} = \mathcal{H}^*$, the set $\{f_m\}$ satisfying the condition

$$\langle e_n, f_m \rangle = \delta_{nm}$$

is also a basis (dual to $\{e_n\}$), and the sets $\{e_n\}$ and $\{f_m\}$ form a biorthogonal system.

A *Riesz basis* $\{x_n\}$ for a Hilbert space $\mathcal{H}$ is a basis obtained from an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ by means of an invertible bounded operator $T : \mathcal{H} \to \mathcal{H}$, i.e. taking $T(e_n) = x_n$ for all n. As a consequence of the Riesz representation theorem, if $\{x_n\}$ is a Riesz basis for a Hilbert space $\mathcal{H}$, there is a unique set $\{y_m\}$ biorthogonal to $\{x_n\}$, so that $\{y_m\}$ is also a Riesz basis. Indeed, for each $x \in \mathcal{H}$ there is a unique sequence of scalars $\{c_n\}$ such that (in the norm sense)

$$x = \sum_{n \in \mathbb{N}} c_n x_n,$$

so that the coefficients define a linear bounded functional

$$f_n : \mathcal{H} \to \mathbb{C} : x \mapsto c_n.$$

On the one hand, if $y \in \mathcal{H}$ is the unique element satisfying $x = T(y)$, and we write $y = \sum_{n \in \mathbb{N}} \langle y, e_n \rangle e_n$, then

$$T(y) = \sum_{n \in \mathbb{N}} \langle y, e_n \rangle T(e_n) = \sum_{n \in \mathbb{N}} c_n x_n,$$

so that $c_n = \langle y, e_n \rangle$ and, since the Cauchy–Schwarz inequality implies that $|\langle y, e_n \rangle| \leq \|T^{-1}\| \|x\|$ and we have that, for every $x \in \mathcal{H}$, its sequence of coefficients $\{c_n\}$ belongs to $\ell^2(\mathbb{N})$. On the other hand, by the Riesz representation theorem, for each $n \in \mathbb{N}$ there exists a unique $y_n \in \mathcal{H}$ such that $f_n(x) = \langle x, y_n \rangle$ for every $x \in \mathcal{H}$. From the unique representation

$$x = \sum_{n \in \mathbb{N}} \langle x, y_n \rangle x_n$$

then follows that $\{y_n\}$ is also a Riesz basis.

Consider a Riesz basis $\{x_n\}$ for a Hilbert space $\mathcal{H}$, and the linear operators

$$\ell_{\{x_n\}} : \mathcal{H} \to \ell^2(\mathbb{N})$$

and

$$H_{\{x_n\}} : \ell^2(\mathbb{N}) \to \mathcal{H}$$

given, respectively, by

$$\ell_{\{x_n\}}(y) = \{\langle y, x_n \rangle\} \quad \text{and} \quad H_{\{x_n\}}(\{c_m\}) = \sum_{n \in \mathbb{N}} c_n x_n.$$

That both $\ell_{\{x_n\}}$ and $H_{\{x_n\}}$ are bounded linear operators follows from the frame inequality for the Riesz basis $\{x_n\}$:

$$A\|x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle|^2 \leq B\|x\|^2,$$

for constants A and B such that $0 \leq A \leq B$. Moreover, these operators are (formal) adjoint to each other and, as a consequence, given a Riesz basis $\{x_n\}$ for a Hilbert space $\mathcal{H}$ we obtain a one to one correspondence between sequences of the form $\{\langle x, x_n \rangle\}$ and sequences in $\ell^2(\mathbb{N})$.

2.2 An Example: Nonharmonic Fourier Series

Let us denote by $\mathbb{S}^1$ the circle in $\mathbb{R}^2$ (seen as the quotient $[-\pi, \pi]/\sim$, where the boundary points in the interval are identified —it is also customary to work with $[0, 1]$ or $[0, 2\pi]$, with the obvious modifications). Consider the Hilbert space $L^2(\mathbb{S}^1)$ of complex-valued functions over the circle, i.e. periodic functions over the interval, with inner product

$$(f, g) = \int_{-\pi}^{\pi} f(x)\bar{g}(x)dx.$$

Recall that a family or sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{S}^1)$ is called an orthogonal family if $(f_i, f_j) = 0$ whenever $i \neq j$, and is called an orthonormal family if, in addition, $(f_i, f_i) = 1$ for any $i \in \mathbb{N}$. An orthonormal basis for $L^2(\mathbb{S}^1)$ is an orthonormal family that spans the whole space. It is easy to see that the functions

$$e_n(x) = e^{inx},$$

for $n \in \mathbb{Z}$, form an orthonormal family[1] in $L^2(\mathbb{S}^1)$.

The nth *Fourier coefficient* of $f \in L^2(\mathbb{S}^1)$ is the complex number

$$\hat{f}(n) = (f, e_n) = \int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

It is easy to see that an orthonormal family is a basis if and only if

$$f = \sum_{n=1}^{\infty} \hat{f}(n)e_n \tag{1}$$

for any $f \in L^2(\mathbb{S}^1)$, and we call (1) the *Fourier series* of f (with respect to the orthonormal basis $\{e_n\}$). A direct computation shows that, for any orthonormal family $\{e_n\}$ and any $f \in L^2(\mathbb{S}^1)$,

$$0 \leq \left\| f - \sum_{n=1}^{N} \hat{f}(n)e_n \right\|^2 = \|f\|^2 - \sum_{n=1}^{N} |\hat{f}(n)|^2,$$

so that $\{e_n\}$ is a basis if and only if *Plancharel identity*

$$\|f\|^2 = \int_0^1 |f(x)|^2 dx = \sum_{n=1}^{\infty} |\hat{f}(n)|^2 \tag{2}$$

holds for all $f \in L^2(\mathbb{S}^1)$.

The map $\mathcal{F} : L^2(\mathbb{S}^1) \to L^2(\mathbb{Z}) : f \mapsto \hat{f}$ defines a isometric isomorphism, called the *Fourier transform* of f, and it actually preserves inner products (Parseval identity) [15]. We will often write $\mathcal{F}$ in compact form as

$$\hat{f}(n) = \int_{\mathbb{S}^1} e^{-inx} f(x)dx,$$

[1] Working with the identification $\mathbb{S}^1 \cong [0, 1]/\sim$ make some authors to work with the orthonormal family $e_n(x) = e^{2\pi inx}$, $n \in \mathbb{Z}$.

choosing an appropriate normalization for the measure dx on the circle, with inverse $\mathcal{F}^{-1}$ given by (1). Recall that one of the main features of the Fourier series is that, for important classes of functions, they can be used to approximate such functions using trigonometric polynomials. Thus, working with Riesz basis instead of the regular orthonormal basis in (1), gives rise to different methods of approximation.

Non-harmonic Fourier series are exponential series of the form

$$\sum_n c_n e^{i\lambda_n x},$$

where $x \in [-\pi, \pi]$ and λ_n are not necessarily integers for all $n \in \mathbb{Z}$. They where studied by Paley and Wiener in [8], and by many other authors from then (see e.g. [19]), and can be thought as perturbations of orthonormal basis. In contrast to what is assumed in harmonic analysis, where there is a *symmetry* in the underlying space (a circle, i.e. we work with e^{inx} with x in $\mathbb{S}^1$ and $n \in \mathbb{Z}$, its dual), in the case of non-harmonic analysis such symmetries in the underlying space do not appear in general. Thus, we can say that harmonic analysis is linked to a background group structure (explicit in both, the group and its dual) while, in the case of non-harmonic analysis, such a background group structure is absent. An important result established by Paley and Wiener, relating Riesz basis with orthonormal basis, is the following [8]:

Theorem 1 *Let $\{e_n\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and suppose that $\{x_n\}$ is a sequence in $\mathcal{H}$ such that, for some $\lambda \in [0, 1)$,*

$$\left\| \sum_n \alpha_n (e_n - x_n) \right\| \leq \lambda \left(\sum_n |\alpha_n|^2 \right)^{\frac{1}{2}},$$

for every finite sequence $\{\alpha_n\}$. Then $\{x_n\}$ is a Riesz basis for $\mathcal{H}$.

From this theorem, and other basic results in functional analysis, many important and interesting properties of these general non-harmonic Fourier series can be established, such as completeness, convergence, stability, etc. For example, from the (usual harmonic) orthonormal basis for $L^2(\mathbb{S}^1)$ given by the functions $e_n(x) = e^{inx}$, for $n \in \mathbb{Z}$, follows that the collection of functions $x_n(x) = e^{i\lambda_n x}$, where $\lambda_n \in \mathbb{R}$ for $n \in \mathbb{Z}$, form a Riesz basis for $L^2(\mathbb{S}^1)$ whenever there exists a constant C such that

$$|\lambda_n - n| \leq C < \frac{1}{\pi^2}$$

for all $n \in \mathbb{Z}$. Generalizations of this result include, for example, the case of more general L^p-spaces and the improvement of the constant $\frac{1}{\pi^2}$ in the last inequality; they can be found in [8, 19].

An alternative point of view on the usual harmonic basis $\{\phi_n = e^{inx}\}_{n\in\mathbb{Z}}$ comes from geometric spectral analysis, where it appears as a system of eigenfunctions for the differential operator

$$D = -i\frac{d}{dx},$$

acting on functions on the interval (0, 1) with periodic boundary conditions. Indeed, on $L^2(0, 1)$ this operator is self-adjoint, and $\{\phi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of periodic eigenfunctions for D. Fourier analysis comes therefore as a decomposition of periodic functions in such a basis, which encodes a particular symmetry through the chosen *periodic boundary condition*. The interplay between the presence of symmetries (encoded in periodic boundary conditions) and the decomposition of functions in terms of bases for Hilbert spaces has important consequences in the study of fundamental physical systems (see, for example, [2] in this volume). In the following sections we will illustrate the use of harmonic analysis to build global pseudo-differential calculus associated to spaces with prescribed symmetries (compact or locally compact Lie groups, homogeneous spaces, etc), on the one hand, or associated to spaces on which such symmetries are absent (through the use of Riesz systems), on the other hand.

In the following, we will work with systems of functions which form Riesz basis for spaces of square-integrable functions. As we have seen, for a complete system of functions $\{\psi_n\}$ in $L^2(0, 1)$ we can weaken the requirement concerning the orthogonality of the system $\{\psi_n\}$, assuming that there exists a complete system of functions $\{g_n\}$ in $L^2(0, 1)$ such that $(\psi_n, g_n) = 1$, $(\psi_n, g_m) = 0$ for all $n \neq m$. In particular, when the system $\{\psi_n\}$ is orthonormal, $g_n = \psi_n$ for all $n \in \mathbb{N}$ (as in the case of usual trigonometric polynomials). If a series

$$\sum_{n=1}^{\infty} a_n \psi_n$$

converges to a function f in $L^2(0, 1)$, then $a_n = (f, g_n)$ for all $n \in \mathbb{N}$. The number $a_n = (f, g_n)$ is called the nth *Fourier coefficient of the function f with respect to the system* $\{\psi_n\}$. Thus, a *Riesz system* is a system $\{\psi_n\}$ satisfying:

(1) For any function f the series of the squares of the Fourier coefficients is absolutely convergent, i.e.

$$\sum_{n=1}^{\infty} |(f, g_n)|^2 < \infty,$$

(2) for any sequence of numbers $\{a_n\} \in \ell^2(\mathbb{N})$ there exists a function f for which the a_n are its Fourier coefficients with respect to the system $\{\psi_n\}$, that is $a_n = (f, g_n)$ for all $n \in \mathbb{N}$.

In practice, (1) is Bessel's inequality and (2) the Riesz–Fischer theorem for Riesz systems. N.K. Bari has proven that a system $\{\psi_n\}$ is a Riesz system if and only if there exists a continuous linear operator T, invertible in $L^2(0, 1)$, such that the

system of functions $\{T(\psi_n)\}$ is complete and orthonormal, i.e. it is equivalent to an orthonormal basis (see [3, 6]).

3 Harmonic Analysis and Pseudo-Differential Operators on Spaces with Symmetries

In this section, we will recall basic facts on the use of the representation theory of compact Lie groups to define a global calcuclus of pseudo-differential operators on that spaces. In particular, we will emphasize the role of the Laplace operator in the constructions of spaces, and other mathematical tools, that allow us to define algebras of pseudo-differential operators modeled on the unitary dual of the compact Lie group, instead of the geometric dual usually used in the construction of local symbols. For a complete account of these methods and results the reader is referred to [12].

3.1 Sobolev Spaces and the Spectrum of the Laplacian

The study of existence and regularity of solutions to partial differential equations led to the use of different Banach and Hilbert spaces of functions (on $\mathbb{R}^n$, Lie groups and manifolds). Sobolev spaces on $\mathbb{R}^n$ are defined as spaces of functions for which such functions and their derivatives (up to some fixed degree) are square integrable: If we denote by $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index,

$$H^k(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \middle| \ \partial_x^\alpha f \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \leq k \right\},$$

where

$$\partial_x^\alpha f = (-i)^{|\alpha|} \frac{\partial^{\alpha_1} f}{\partial {x_1}^{\alpha_1}} \frac{\partial^{\alpha_2} f}{\partial {x_2}^{\alpha_2}} \cdots \frac{\partial^{\alpha_n} f}{\partial {x_n}^{\alpha_n}},$$

and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. In the definition before $k \in \mathbb{N}$, but Sobolev spaces have a good behavior with respect to Fourier transform that led to the notion of $H^s(\mathbb{R}^n)$ for real order $s > 0$, and for $s < 0$ as a space of distributions (for $s = 0$ the Sobolev space is the Hilbert space $L^2(\mathbb{R}^n)$). Indeed, if we consider the (Fourier) dual variable $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, and we denote by $\langle \xi \rangle = (1 + \|\xi\|^2)^{\frac{1}{2}}$, for any $k \in \mathbb{N}$ and any $f \in L^2(\mathbb{R}^n)$, the equivalence $f \in H^k(\mathbb{R}^n) \Leftrightarrow \langle \xi \rangle^k \mathcal{F}(f) \in L^2(\mathbb{R}^n)$ shows that, for general $s \in \mathbb{R}$,

$$\begin{aligned} H^s(\mathbb{R}^n) &= \left\{ f \in S'(\mathbb{R}^n) \middle| \ \langle \xi \rangle^s \mathcal{F}(f) \in L^2(\mathbb{R}^n) \right\} \\ &= \left\{ f \in S'(\mathbb{R}^n) \middle| \ \mathcal{F}^{-1}\left(\langle \xi \rangle^s \mathcal{F}(f) \right) \in L^2(\mathbb{R}^n) \right\}, \end{aligned}$$

where $S'(\mathbb{R}^n)$ denotes the space of tempered distributions (continuous linear functionals on Schwartz functions), $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively, in $\mathbb{R}^n$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \tag{3}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ denotes the dual variable (the duality realized through the inner product in $\mathbb{R}^n$, and the integral conveniently normalized).

From the inner product in $L^2(\mathbb{R}^n)$, to each $H^s(\mathbb{R}^n)$ can be given the structure of Hilbert space, using the inner product

$$(f, g)_{H^s(\mathbb{R}^n)} = \left(\mathcal{F}^{-1}\left(\langle\xi\rangle^s \mathcal{F}(f)\right), \mathcal{F}^{-1}\left(\langle\xi\rangle^s \mathcal{F}(g)\right)\right)_{L^2(\mathbb{R}^n)}.$$

Notice that, for $s < 0$, the spaces $H^s(\mathbb{R}^n)$ are spaces of distributions, and the dual space to $H^s(\mathbb{R}^n)$ is precisely $H^{-s}(\mathbb{R}^n)$. The invariance under coordinate transformations permits an invariant notion of $H^s(M)$ for a compact manifold M, with or without boundary (see e.g. [16, 17]). It is worth to mention that, on a compact Riemannian manifold (M, g), the Sobolev space $H^k(M)$ carry important spectral information encoded by the Laplace-Beltrami operator associated to the Riemannian metric on M. Indeed, if Δ_g denotes the Laplace-Beltrami operator associated to the metric g, $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ denote the eigenvalues of Δ_g (for which $\lambda_n \to \infty$ as $n \to \infty$) and $\{\varphi_n\}$ denote the associated eigenfunctions, these eigenfunctions form a complete orthonormal system for $L^2(M)$ and, for any $f \in L^2(M)$, we have a Fourier series expansion of the form

$$f = \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n,$$

where the coefficients $a_n = (f, \varphi_n)_{L^2(\mathbb{R}^n)}$ satisfy $\sum_{n=1}^{\infty} a_n^2 = \|f\|_{L^2(\mathbb{R}^n)} < \infty$. Moreover, the Sobolev space $H^k(M)$ can be defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|f\|_{H^k(M)} = \|f\|_{H^{k-1}(M)} + \int_M \langle \nabla^k f, \nabla^k f\rangle_M \, d\mu_g,$$

where ∇^k is the iterated covariant derivative with respect to the Levi-Civita connection on M and $d\mu_g$ is the Borel measure induced by the volume form (recall that $H^0(M) = L^2(M)$). In that case, it can be shown that [4]

$$H^1(M) = \left\{ f = \sum_{n=0}^{\infty} a_n\varphi_n \;\middle|\; \sum_{n=0}^{\infty} \lambda_n a_n^2 < \infty \right\}, \tag{4}$$

and a similar result holds true for higher order Sobolev spaces. Thus, in general, the function spaces we work with can be characterized by the behavior of Fourier coefficients against the spectrum of the Laplace–Beltrami operator, which appears here as a reference operator in alternative definitions for norms and inner products in function spaces. This feature will be widely used in the following sections.

3.2 Pseudo-Differential Operators

Pseudo differential operators are very powerful tools for solving linear partial differential equations. They are, in general, integral operators defined using the usual Fourier integral representation of derivatives applied to more general functions (called symbols) which, from a theoretical-physics perspective, correspond to the classical limit of the (quantum) operators associated to observables.

3.2.1 Local Symbols and Pseudo-Differential Operators

A *pseudo-differential operator* P, defined by a local symbol σ_P on (rapidly decaying) functions in $\mathbb{R}^n$, is an expression of the form [7]:

$$Pf(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_P(x,\xi)\hat{f}(\xi)d\xi, \tag{5}$$

where the *symbol* $\sigma_P(x,\xi) \in S^m(\mathbb{R}^n)$ if and only if it is smooth in $\mathbb{R}^n \times \mathbb{R}^n$ and there exists a constant $C_{\alpha,\beta}$ such that (on any compact set)

$$|\partial_x^\beta \partial_\xi^\alpha \sigma_P(x,\xi)| \leq C_{\alpha,\beta}(1+\|\xi\|)^{m-|\alpha|},$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$ and any $x, \xi \in \mathbb{R}^n$. Here $\hat{f}(\xi)$ denotes the Fourier transform of f, given by (3), and the integral expression can be understood using the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)d\xi. \tag{6}$$

Elements of $S^m(\mathbb{R}^n)$ are called (local) symbols of order $m \in \mathbb{R}$, and they give rise through (5) to pseudo-differential operators of order m, the set of which will be denoted by $\Psi^m(\mathbb{R}^n)$. Important examples of pseudo-differential operators include the following:

1. Differential operators of the form

$$D = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha,$$

where the coefficients a_α are smooth functions on $\mathbb{R}^n$. It follows that differential operators of degree m are pseudo-differential operators with polynomial local symbol $\sigma(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$, as follows from Fourier inversion formula and the basic properties of the Fourier transform:

$$Df(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha(f) = \sum_{|\alpha|\leq m} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_\alpha(x)\xi^\alpha \hat{f}(\xi)d\xi.$$

2. The inverse Δ^{-1} of the Laplace operator $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^n$ (which is a second order differential operator with constant coefficients) is a pseudo-differential operator of degree -2.
3. The heat kernel operator $e^{-t\Delta}$, for $t > 0$, is a smoothing pseudo-differential operator (i.e. a pseudo-differential operator of degree $-\infty$).

These examples include pseudo-differential operators commonly used in (linear and nonlinear) theory of partial differential equations, global analysis and mathematical physics. As mentioned before, pseudo-differential operators can be defined (using local charts) on manifolds and vector bundles (with matrix coefficients), using a local definition of the corresponding symbol classes, and dealing with the dependence of the structure on the local charts (see e.g. [16]). It is customary to see, in the more general case, the local symbols as sections of a vector bundle over the cotangent space (i.e. to see $\sigma_P(x,\xi)$ as a matrix, when evaluated in $(x,\xi) \in T_x^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, using local charts). We will only use these expressions as a reference to give a hint about the use of a different dual (to the geometric dual), the unitary dual of a compact Lie group, in a construction giving rise to symbols with values in representation spaces of the Lie group in the following section. Pseudo-differential operators can be also defined using distributional kernels, as it is done in [14], where more properties and uses of pseudo-differential operators on smooth manifolds can be found.

3.2.2 Matrix Symbols and Global Pseudo-Differential Operators on Lie Groups

Let us denote by G a compact Lie group, then each irreducible representation of G is finite dimensional and every unitary representation of G is a direct sum of irreducible representations. We will denote by $\widehat{G}$ the set of all unitary equivalence classes of irreducible unitary representations of G, called the unitary dual of G,

$$\widehat{G} := \{[\xi] \mid \xi : G \xrightarrow{\text{irrep}} \mathsf{U}(\mathcal{H}_\xi)\}.$$

If ξ is any unitary representation of G in the complex Hilbert space $(\mathcal{H}_\xi, \langle\cdot,\cdot\rangle_{\mathcal{H}_\xi})$, the matrix elements of ξ, i.e. the functions $\xi_{ij}(x) = \langle \xi(x)e_i, e_j\rangle_{\mathcal{H}_\xi}$, for an orthonormal basis $\{e_i\}_{i=1}^{\dim \mathcal{H}_\xi}$ for $\mathcal{H}_\xi$, generate the space of trigonometric polynomials of ξ. The *Peter-Weyl theorem* guarantees that the union over all elements in $\widehat{G}$ of the spaces

of trigonometric polynomials is uniformly dense in $C(G)$. As a consequence, *acting on irreducible unitary representations of G is enough to act on any continuous (or smooth, or L^2) function on G*. Moreover, since for any compact Lie group G the unitary dual $\widehat{G}$ turns out to be *discrete*, this amounts to have a discrete expression for *any linear operator*

$$A : C(G) \to C(G).$$

Objects realizing this discretization are called *global symbols* (see [12]).

Let us, following [12], recall the definition of the global symbol of the continuous linear operator A. The Fourier components of a function $\varphi \in L^2(G)$ can be defined in terms of the unitary dual $\widehat{G}$ of G as follows. Given an irreducible unitary representation $\xi \in \widehat{G}$, consider the map

$$\xi : G \to \mathbb{C}^{d_\xi \times d_\xi} : x \mapsto [\xi_{ij}(x)]$$

defined by its matrix coefficients. Then, the Peter-Weyl theorem gives an orthonormal basis $\{\sqrt{d_\xi}\xi_{ij}\}_{1\leq i,j\leq d_\xi}$ for $L^2(G)$, where we denote by d_ξ the dimension of the Hilbert space $\mathcal{H}_\xi$. Furthermore, using such orthonormal basis for $L^2(G)$, the Fourier components of any $\varphi \in L^2(G)$ is defined as the matrix-valued function $\widehat{\varphi} : \widehat{G} \to \mathbb{C}^{d_\xi \times d_\xi}$ given by

$$\widehat{\varphi}(\xi) = \int_G \varphi(x)\xi(x)^* dx,$$

where dx denotes the Haar measure on G. Both, the following Fourier inversion formula

$$\varphi(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\widehat{\varphi}(\xi)),$$

and the Plancherel identity on $L^2(G)$

$$\|\varphi\|_{L^2(G)} = \left(\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\widehat{\varphi}(\xi)\widehat{\varphi}(\xi)^*)\right)^{\frac{1}{2}} = \|\widehat{\varphi}\|_{\ell^2(\widehat{G})},$$

where the term within the sum is the Hilbert-Schmidt norm of the matrix $\widehat{\varphi}(\xi)$, follow easily from the definitions and properties of the Haar measure on G. In the case $G = S^1$ this is exactly Fourier analysis for periodic functions as given by (1) and (2).

Now, this means that we can associate to any linear continuous operator $A : C^\infty(G) \to C^\infty(G)$ a matrix-valued or *full symbol* $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ given by

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x), \tag{7}$$

which must be understood from the action of the operator on functions using Fourier decomposition $f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\widehat{f}(\xi))$, i.e.

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)]. \tag{8}$$

Such operators are called *global pseudo-differential operators*, and their algebraic and analytic properties have been systematically studied in [12].

Pseudo-differential operators (8) defined by matrix symbols (7) differ from (5) by the nature of their symbols. In the local case $\sigma_P(x,\xi) \in T^*\mathbb{R}^n$ (or T^*U for $U \subset \mathbb{R}^n$) while, in the case of the compact Lie group G, instead of using functions on T^*G as symbols we can use, via Fourier analysis, elements in $G \times \widehat{G}$. This theory, called *global pseudo-differential calculus*, has shown to be useful to "discretize" half of the variables, replacing integrals by sums in calculations. In contrast with the local symbols $\sigma(x,\xi)$ used in the context of classical (pseudo-differential) operator theory, the moment variables in the matrix-valued symbols $\sigma_A(x,\xi)$ considered before are *discrete* variables. In particular, for example, derivatives and other (more involved) operators can be seen as (series of) matrices. Natural examples of matrix symbols of global pseudo-differential operators include the following:

1. For $A = I$, the identity, it follows that

$$\sigma_I(x,\xi) = \xi(x)^*\xi(x) = I_{d_\xi\times d_\xi}$$

   is its matrix symbol.
2. If $A = \mathcal{L}_G$ is the Laplacian on G, it can be diagonalized in the basis given by the Peter-Weyl theorem for $L^2(G)$, and it follows that its symbol is given by

$$\sigma_{\mathcal{L}_G}(x,\xi) = \xi(x)^*\mathcal{L}_G\xi(x) = -\lambda_{[\xi]}I_{d_\xi\times d_\xi}.$$

The Laplace operator $\mathcal{L}_G$ on the compact Lie group G gives us a natural way to compare the behavior, under derivatives of symbols in the local classes defined in the last section (the Hörmander classes $\Psi^m(G)$), with the behavior under discrete difference operations of matrix symbols. Indeed, since for $\xi \in \widehat{G}$ and $x \in G$ fixed, there exists a non-negative real number $\lambda_{[\xi]}$ such that

$$\mathcal{L}_G\xi(x) = -\lambda_{[\xi]}\xi(x),$$

where $\mathcal{L}_G$ is the Laplacian on the group G, letting $\langle\xi\rangle := (1+\lambda_{[\xi]})^{\frac{1}{2}}$ we obtain the following characterization of the Hörmander classes $\Psi^m(G)$ (as defined by local symbols on manifolds) [12]:

Theorem 2 *A linear operator $A \in \Psi^m(G)$ if and only if its matrix-valued symbol $\sigma_A(x,\xi)$ satisfies the inequalities*

$$\|\partial_x^\alpha \mathbb{D}^\beta \sigma_A(x,\xi)\|_{op} \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}, \tag{9}$$

for every $\alpha, \beta \in \mathbb{N}^n$. *Here* $\mathbb{D}^\beta$ *denote difference operators in the discrete variables.*

The difference operators used in the theorem before can be defined using as generators matrix-valued differences corresponding to symbols of the type $\xi(g) - \mathrm{I}$ (see [13] and references therein for more details, the reader can also go to Sect. 4.6 to see a generalization used in a much more broad context). In the case of the circle S^1, for example, since all irreducible representations are one-dimensional and can be parametrized by integers, and therefore $\widehat{S^1} \cong \mathbb{Z}$, for each $n \in \mathbb{Z}$ we take $\xi_n(\theta) = e^{in\theta}$, the Fourier transform of a function is

$$\hat{f}(n) = \int_{\mathbb{S}^1} f(\theta) e^{-in\theta} d\theta,$$

so that global pseudo-differential operators on $\mathbb{S}^1$ have matrix symbols $\sigma : \mathbb{S}^1 \times \mathbb{Z} \to \mathbb{C}$ from which we can write

$$A_\sigma f(\theta) = \sum_{n\in\mathbb{Z}} \sigma(\theta, n)\hat{f}(n)e^{in\theta},$$

with the obvious notations. In this case, the Hörmander classes correspond to global classes

$$S^m(\mathbb{S}^1) = \{\sigma : \|\Delta^\alpha \partial_\theta^\beta \sigma(\theta, n)\| \leq C_{\alpha,\beta}(1 + |n|^2)^{\frac{1}{2}(m-\alpha)}\},$$

where Δ is the difference operator acting on functions by $\Delta g(n) := g(n+1) - g(n)$.

Non-abelian examples involve series of matrices increasing in size, but are equally treatable. For example, if $G = SU(2)$, since $\widehat{SU(2)} \cong \frac{\mathbb{N}}{2}$, and the irreducible unitary representations of this group are given, for $\ell \in \frac{1}{2}\mathbb{N}$, by matrix multiplication on spaces of homogeneous polynomials $V_\ell = \{f \in \mathbb{C}[z_1, z_2] : \sum_{k=0}^{2\ell} a_k z_1^k z_2^{2\ell-k}, a_k \in \mathbb{C}\}$, let us denote by $T^\ell : SU(2) \to \mathsf{U}(V_\ell)$ these representations. Any linear operator $A : C^\infty(SU(2)) \to C^\infty(SU(2))$ defines a global pseudo-differential operator with matrix symbol $\sigma : SU(2) \times \frac{\mathbb{N}}{2} \to GL(V_\ell)$, where the discrete part on the domain models the dual (momentum) space for $SU(2)$. The corresponding global pseudo-differential operator $A_\sigma : C^\infty(SU(2)) \to C^\infty(SU(2))$ can be written as

$$A_\sigma f(x) := \sum_{\frac{1}{2}\mathbb{N}} (2\ell+1)\mathrm{Tr}(T^\ell(x)\sigma(x,\ell)\hat{f}(\ell)),$$

where the Fourier transform of the function f is the matrix-valued function $\hat{f}(\ell) = \int_{SU(2)} f(y)T^\ell(y)^* dy$. In particular, for example, in these variables the Laplacian $\mathcal{L}$ on $SU(2)$ has as matrix symbol

$$\sigma_{\mathcal{L}}(x,\ell) = -\ell(\ell+1)\mathbb{I}_{2\ell+1\times 2\ell+1}.$$

Remark 1 In order to do explicit computations associated with representations, it is necessary to know all the entries of the matrix representations described above with respect to the given basis (dual variables), in terms of the parameters used to describe the group elements (e.g. generalized Euler angles). Regarding particular applications, it is convenient to have an indexation of the sum in terms of geometric parametrizations of the unitary dual (highest weights, coadjoint orbits, etc). These methods have been used to introduce a natural classes of pseudo-differential operators on more general algebraic objects, such as compact quantum groups, where Fourier analysis and distribution theory can be developed [1].

In the monograph [12] the reader can find many results on the algebra of global pseudo-differential operators on compact Lie groups, and extensive references to explicit formulae and applications. We could go on, for example, with the conventional results on composition and asymptotic formulae for symbols of adjoint operators, but we will postpone those to formulate them in much greater generality in the last section, devoted to the non-harmonic pseudo-differential analysis of boundary value problems.

4 Nonharmonic Analysis of Boundary Value Problems

In this section we reach the goal of this chapter, giving an introduction to what will be called *non-harmonic analysis of boundary value problems*. Throughout this section we will work on a domain Ω with boundary $\partial\Omega$, using a model operator $\mathfrak{L}$ (which will play the role the Laplace operator played already in the last sections). The domain Ω can be a manifold with boundary or a domain in $\mathbb{R}^n$, and the model operator $\mathfrak{L}$ can be a differential (or pseudo-differential) operator defined on spaces of square-integrable functions on Ω, and satisfying boundary conditions on $\partial\Omega$. Here we will assume that $\Omega \subset \mathbb{R}^n$, but a complete account of the non-harmonic analysis of boundary value problems on a manifold can be found in [5, 10, 11].

4.1 Basic Assumptions

We will be working with discrete sets of eigenvalues and eigenfunctions indexed by a countable set $\mathcal{I} \subset \mathbb{N} \cup \{0\}$, and make use of two special assumptions regarding the pair $(\Omega, \mathfrak{L})$ (and its associated adjoint $(\Omega, \mathfrak{L}^*)$).

Assumption 1. Let Ω be a smooth n-dimensional manifold with boundary $\partial\Omega$, such that $\bar{\Omega}$ is a compact manifold, not necessarily smooth in the boundary. Consider a differential operator $\mathfrak{L}$ of order m with smooth bounded coefficients in Ω, equipped with some fixed linear boundary conditions. Assume that the boundary conditions lead to a discrete spectrum of $\mathfrak{L}$ with a family of eigenfunctions yielding a Riesz

basis in $L^2(\Omega)$. The discrete sets of eigenvalues and eigenfunctions will be indexed by the countable set $\mathcal{I}$ and, without loss of generality, will be always a subset of $\mathbb{Z}^k$ for some $k \in \mathbb{N}$. We consider the spectrum $\{\lambda_\xi \in \mathbb{C} : \xi \in \mathcal{I}\}$ of $\mathfrak{L}$ with corresponding eigenfunctions in $L^2(\Omega)$ denoted by u_ξ, i.e.

$$\mathfrak{L}u_\xi = \lambda_\xi u_\xi \text{ in } \Omega, \quad \text{for all } \xi \in \mathcal{I},$$

and the eigenfunctions u_ξ satisfy the fixed boundary conditions (BC). The conjugate spectral problem is

$$\mathfrak{L}^* v_\xi = \overline{\lambda_\xi} v_\xi \text{ in } \Omega, \quad \text{for all } \xi \in \mathcal{I},$$

which we equip with the conjugate boundary conditions, denoted by $(\mathrm{BC})^*$. We also assume that the functions u_ξ, v_ξ are normalized, i.e. $\|u_\xi\|_{L^2(\Omega)} = 1$ and $\|v_\xi\|_{L^2(\Omega)} = 1$, for all $\xi \in \mathcal{I}$, and there exist a constant $C > 0$ and $\mu_0 > 0$ such that

$$\sup_{x\in\Omega} |u_\xi(x)| \leq C\langle\xi\rangle^{\mu_0}$$

for every $\xi \in \mathcal{I}$, where

$$\langle\xi\rangle := (1 + |\lambda_\xi|^2)^{\frac{1}{2m}},$$

and m is the order of the differential operator $\mathfrak{L}$. The systems $\{u_\xi\}_{\xi\in\mathcal{I}}$ and $\{v_\xi\}_{\xi\in\mathcal{I}}$ are biorthogonal with respect to the usual inner product on the Hilbert space $L^2(\Omega)$ i.e.,

$$\langle u_\xi, v_\eta\rangle_{L^2(\Omega)} = \int_\Omega u_\xi(x)\overline{v_\eta(x)}dx = \delta_{\xi\eta}.$$

Assumption 2. The system $\{u_\xi\}_{\xi\in\mathcal{I}}$ is a basis for $L^2(\Omega)$.

4.2 Test Functions for $\mathfrak{L}$ and $\mathfrak{L}^*$

We can associate spaces of distributions to the differential operator $\mathfrak{L}$ and its adjoint $\mathfrak{L}^*$ as follows (see [10, 11]).

Definition 1 We define the space of *test functions for the differential operator* $\mathfrak{L}$ as

$$C^\infty_{\mathfrak{L}}(\overline{\Omega}) := \mathrm{Dom}\left(\mathfrak{L}^\infty\right) := \bigcap_{k=1}^{\infty} \mathrm{Dom}\left(\mathfrak{L}^k\right),$$

where $\mathrm{Dom}\left(\mathfrak{L}^k\right)$ is the domain of $\mathfrak{L}^k$. Here

$$\mathrm{Dom}\left(\mathfrak{L}^k\right) := \left\{f \in L^2(\Omega) : \mathfrak{L}^j f \in L^2(\Omega), \text{for } j = 0, 1, 2, \ldots, k-1\right\}$$

and the boundary conditions (BC) are satisfied by each one of the operators $\mathfrak{L}^j$, for $j = 0, 1, \cdots, k-1$.

The space $C^\infty_{\mathfrak{L}^*}(\Omega)$ of *test functions for the adjoint operator* $\mathfrak{L}^*$ is the given by

$$C^\infty_{\mathfrak{L}^*}(\overline{\Omega}) := \mathrm{Dom}\left((\mathfrak{L}^*)^\infty\right) := \bigcap_{k=1}^{\infty} \mathrm{Dom}\left(\left(\mathfrak{L}^*\right)^k\right),$$

where $\mathrm{Dom}\left((\mathfrak{L}^*)^k\right)$ is the domain of the operator $(\mathfrak{L}^*)^k$, given by

$$\mathrm{Dom}\left(\left(\mathfrak{L}^*\right)^k\right) := \left\{ f \in L^2(\Omega) : \left(\mathfrak{L}^*\right)^j f \in L^2(\Omega), \text{for } j = 0, 1, 2, \ldots, k-1 \right\},$$

and each operator $(\mathfrak{L}^*)^j$ satisfies the adjoint boundary conditions (i.e. the corresponding to the operator $\mathfrak{L}^*$), for $j = 0, 1, \cdots, k-1$.

Let us give a Fréchet topology to $C^\infty_{\mathfrak{L}}(\overline{\Omega})$, using the family of norms

$$\|\varphi\|_{C^k_{\mathfrak{L}}} := \max_{j \le k} \left\| \mathfrak{L}^j \varphi \right\|_{L^2(\Omega)}, \quad k \in \mathbb{N}_0,$$

and, similarly, for $C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$ the Fréchet topology given by the family of norms

$$\|\varphi\|_{C^k_{\mathfrak{L}^*}} := \max_{j \le k} \left\| \left(\mathfrak{L}^*\right)^j \varphi \right\|_{L^2(\Omega)}, \quad k \in \mathbb{N}_0.$$

Assumption 1 will be used to ensure the spaces $C^\infty_{\mathfrak{L}}(\overline{\Omega})$ and $C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$ are dense in $L^2(\Omega)$, which follows from the fact that $u_\xi \in C^\infty_{\mathfrak{L}}(\overline{\Omega})$ and $v_\xi \in C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$, for all $\xi \in \mathcal{I}$.

The dual spaces $\mathcal{D}'_{\mathfrak{L}}(\Omega) := \mathcal{L}(C^\infty_{\mathfrak{L}^*}(\overline{\Omega}), \mathbb{C})$ and $\mathcal{D}'_{\mathfrak{L}^*}(\Omega) := \mathcal{L}(C^\infty_{\mathfrak{L}}(\overline{\Omega}), \mathbb{C})$, of linear continuous functionals on $C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$ and $C^\infty_{\mathfrak{L}}(\overline{\Omega})$, respectively, will be called the spaces of $\mathfrak{L}$-*distributions* and $\mathfrak{L}^*$-*distributions*, respectively. Moreover, we will work with the space $\mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(\mathcal{I})$, i.e. the collection of all functions $f : \Omega \times \mathcal{I} \to \mathbb{C}$ such that $f(x, \cdot) \in \mathcal{S}'(\mathcal{I})$, for every $x \in \Omega$, and $f(\cdot, \xi) \in \mathcal{D}'_{\mathfrak{L}}(\Omega)$, for every $\xi \in \mathcal{I}$.

4.3 $\mathfrak{L}$-Fourier Transform and $\mathfrak{L}^*$-Fourier Transform

Associated to a pair $(\Omega, \mathfrak{L})$ (and its associated adjoint $(\Omega, \mathfrak{L}^*)$) we can define generalized Fourier transforms, here we recall the definition and some important properties of the $\mathfrak{L}$-Fourier transform (and the $\mathfrak{L}^*$-Fourier transform) associated to the boundary value problem $\mathfrak{L}$, for further properties the reader is referred to [10, 11]. Let $\mathcal{S}(\mathcal{I})$ denote the space of rapidly decaying functions $\phi : \mathcal{I} \to \mathbb{C}$, i.e. $\phi \in \mathcal{S}(\mathcal{I})$ if, for every $\ell < \infty$, there exists a constant $C_{\phi,\ell}$ such that

$$|\phi(\xi)| \le C_{\phi,\ell} \langle \xi \rangle^{-\ell}$$

for all $\xi \in \mathcal{I}$. The topology on $\mathcal{S}(\mathcal{I})$ is given the by the seminorms p_k, for $k \in \mathbb{N}_0$, given by

$$p_k(\phi) := \sup_{\xi \in \mathcal{I}} \langle \xi \rangle^k |\phi(\xi)|.$$

Thus, the continuous linear functionals on $\mathcal{S}(\mathcal{I})$ are functionals of the form

$$\phi \mapsto \langle u, \phi \rangle = \sum_{\xi \in \mathcal{I}} u(\xi)\phi(\xi),$$

where the map $u : \mathcal{I} \to \mathbb{C}$ has the property that it grows at most polynomially at infinity, i.e. there exist constants $\ell < \infty$ and $C_{u,\ell}$ such that

$$|u(\xi)| \leq C_{u,\ell} \langle \xi \rangle^{\ell}$$

for all $\xi \in \mathcal{I}$. Such kind of distributions $u : \mathcal{I} \to \mathbb{C}$ form a distribution space which we denote by $\mathcal{S}'(\mathcal{I})$. we can now define the $\mathfrak{L}$-Fourier transform on $C^{\infty}_{\mathfrak{L}}(M)$ and study its main properties.

Definition 2 The $\mathfrak{L}$-*Fourier transform* is the map

$$(\mathcal{F}_{\mathfrak{L}} f)(\xi) = (f \mapsto \widehat{f}) : C^{\infty}_{\mathfrak{L}}(\overline{\Omega}) \to \mathcal{S}(\mathcal{I})$$

given by

$$\hat{f}(\xi) := (\mathcal{F}_{\mathfrak{L}} f)(\xi) = \int_{\Omega} f(x)\overline{v_{\xi}(x)}\, dx.$$

Analogously, the $\mathfrak{L}^*$-*Fourier transform*

$$(\mathcal{F}_{\mathfrak{L}^*} f)(\xi) = \left(f \mapsto \widehat{f_*}\right) : C^{\infty}_{\mathfrak{L}^*}(\overline{\Omega}) \to S(\mathcal{I})$$

is the map defined by

$$\hat{f}_*(\xi) := (\mathcal{F}_{\mathfrak{L}^*} f)(\xi) = \int_{\Omega} f(x)\overline{u_{\xi}(x)}\, dx.$$

Notice that these definitions give a natural generalization of the corresponding notions in the case of compact Lie groups, whenever we use the Laplace operator $\mathfrak{L}_G$. Moreover, many important properties of the Fourier transform translate naturally to this much richer context, among them the following.

Proposition 1 *Consider the boundary value problem* $(\Omega, \mathfrak{L})$ *(and its associated adjoint* $(\Omega, \mathfrak{L}^*)$*), satisfying* Assumption 1 *and* Assumption 2 *introduced before.*

1. *The* $\mathfrak{L}$*-Fourier transform* $\mathcal{F}_{\mathfrak{L}}$ *is a bijective homeomorphism from* $C^{\infty}_{\mathfrak{L}}(\overline{\Omega})$ *into* $\mathcal{S}(\mathcal{I})$*, with inverse*

$$\mathcal{F}^{-1}_{\mathfrak{L}} : \mathcal{S}(\mathcal{I}) \to C^{\infty}_{\mathfrak{L}}(\overline{\Omega})$$

given by

$$\left(\mathcal{F}_{\mathfrak{L}}^{-1}h\right)(x) = \sum_{\xi\in\mathcal{I}} h(\xi)u_\xi(x), \qquad h \in \mathcal{S}(\mathcal{I}).$$

Moreover, the bijective homeomorphism $\mathcal{F}_{\mathfrak{L}}$ and its inverse give rise to the following Fourier inversion formula:

$$f(x) = \sum_{\xi\in\mathcal{I}} \hat{f}(\xi)u_\xi(x),$$

for all $f \in C^\infty_{\mathfrak{L}}(\overline{\Omega})$.

2. *Similarly, the $\mathfrak{L}^*$-Fourier transform $\mathcal{F}_{\mathfrak{L}^*} : C^\infty_{\mathfrak{L}^*}(\overline{\Omega}) \to \mathcal{S}(\mathcal{I})$ is a bijective homeomorphism with inverse*

$$\mathcal{F}_{\mathfrak{L}^*}^{-1} : \mathcal{S}(\mathcal{I}) \to C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$$

given by

$$\left(\mathcal{F}_{\mathfrak{L}^*}^{-1}h\right)(x) = \sum_{\xi\in\mathcal{I}} h(\xi)v_\xi(x), \qquad h \in S(\mathcal{I}).$$

There is a conjugate Fourier inversion formula given by:

$$f(x) = \sum_{\xi\in\mathcal{I}} \widehat{f}_*(\xi)v_\xi(x),$$

for all $f \in C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$.

Remark 2 Using the results mentioned in Sect. 2.1 (see also [3]), since we work with Riesz bases u_ξ and v_ξ, we can easily establish the following relation between L^2-functions and their Fourier coefficients. Let $f \in L^2(\Omega)$, then there are constants m_1 and m_2 such that

$$m_1^2\|f\|_{L^2}^2 \le \sum_{\xi\in\mathcal{I}} |\hat{f}(\xi)|^2 \le m_2^2\|f\|_{L^2}^2$$

and

$$m_1^2\|f\|_{L^2}^2 \le \sum_{\xi\in I} \left|\hat{f}_*(\xi)\right|^2 \le m_2^2\|f\|_{L^2}^2.$$

4.4 Plancherel Formula and Sobolev Spaces Associated to $\mathfrak{L}$ and $\mathfrak{L}^*$

As the reader presumably noticed, we have already all the elements we need to define Sobolev spaces associated to the model operator $\mathfrak{L}$, on the lines we did mention in

Sect. 3.1. The main ingredient will be the definition of convolutions associated to $\mathfrak{L}$ and $\mathfrak{L}^*$, using the Fourier transforms associated to those operators. We will also recall the Plancherel identities obtained on suitable sequence spaces $\ell^2(\mathfrak{L})$ and $\ell^2(\mathfrak{L}^*)$, respectively, and Parseval identities corresponding to them, for which proofs can be found in references [10, 11].

Definition 3 The $\mathfrak{L}$-*convolution* between f_1 and f_2 in $C^\infty_{\mathfrak{L}}(\overline{\Omega})$ is defined (using the $\mathfrak{L}^*$-Fourier transform) by

$$(f_1 *_{\mathfrak{L}} f_2)(x) := \sum_{\xi\in\mathcal{I}} \widehat{f_1}(\xi)\widehat{f_2}(\xi)u_\xi(x).$$

Analogously, for $f_1, f_2 \in C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$, the $\mathfrak{L}^*$-*convolution* (using the $\mathfrak{L}^*$-Fourier transform) is given by

$$(f_1 *_{\mathfrak{L}^*} f_2)(x) := \sum_{\xi\in\mathcal{I}} \widehat{f_{1*}}(\xi)\widehat{f_{2*}}(\xi)v_\xi(x).$$

Direct computations, from the definitions given before give us back, the usual properties of convolutions under Fourier transforms, for example:

Proposition 2 *For any* $f_1, f_2 \in C^\infty_{\mathfrak{L}}(\overline{\Omega})$,

$$\widehat{f_1 *_{\mathfrak{L}} f_2} = \widehat{f_1}\widehat{f_2}.$$

Analogously, for any $f_1, f_2 \in C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$,

$$\widehat{f_1 *_{\mathfrak{L}^*} f_2} = \widehat{f_{1*}}\widehat{f_{2*}}.$$

In order to define sequence spaces $\ell^2(\mathfrak{L})$ and $\ell^2(\mathfrak{L}^*)$, the generalization of the discrete spaces $\ell^2(\widehat{G})$ used on the unitary dual of a compact Lie group G, we can use the bijective homeomorphism from $C^\infty_{\mathfrak{L}}(\overline{\Omega})$ into $\mathcal{S}(\mathcal{I})$ given by the $\mathfrak{L}$-Fourier transform and the $\mathfrak{L}^*$-Fourier transform, respectively.

Definition 4 Let $\ell^2(\mathfrak{L})$ be the linear space of complex-valued functions g on $\mathcal{I}$ such that $\mathcal{F}^{-1}_{\mathfrak{L}} g \in L^2(\Omega)$, i.e. $g \in \ell^2(\mathfrak{L})$ if and only if there exists $f \in L^2(\Omega)$ such that $\mathcal{F}_{\mathfrak{L}} f = g$. Then, the space of sequences $\ell^2(\mathfrak{L})$ is a Hilbert space with the inner product

$$\langle a, b\rangle_{\ell^2(\mathfrak{L})} := \sum_{\xi\in\mathcal{I}} a(\xi)\overline{\left(\mathcal{F}_{\mathfrak{L}^*} \circ \mathcal{F}^{-1}_{\mathfrak{L}} b\right)(\xi)}$$

for $a, b \in \ell^2(\mathfrak{L})$. Analogously, the Hilbert space $\ell^2(\mathfrak{L}^*)$ is the linear space of complex-valued functions g on $\mathcal{I}$ such that $\mathcal{F}^{-1}_{\mathfrak{L}^*} g \in L^2(\Omega)$, with inner product

$$\langle a, b\rangle_{\ell^2(\mathfrak{L}^*)} := \sum_{\xi\in\mathcal{I}} a(\xi)\overline{\left(\mathcal{F}_{\mathfrak{L}} \circ \mathcal{F}^{-1}_{\mathfrak{L}^*} b\right)(\xi)}$$

for $a, b \in \ell^2(\mathfrak{L}^*)$.

The norm on $\ell^2(\mathfrak{L})$ is given by

$$\|a\|_{\ell^2(\mathfrak{L})} = \sum_{\xi\in\mathcal{I}} a(\xi)\overline{\left(\mathcal{F}_{\mathfrak{L}^*}\circ\mathcal{F}_{\mathfrak{L}}^{-1}a\right)(\xi)}, \qquad a \in \ell^2(\mathfrak{L}),$$

and an analogue formula exists for $\ell^2(\mathfrak{L}^*)$. Notice that the spaces $\ell^2(\mathfrak{L})$ and $\ell^2(\mathfrak{L}^*)$ are generated by the biorthogonal systems $\{u_\xi\}_{\xi\in\mathcal{I}}$ and $\{v_\xi\}_{\xi\in\mathcal{I}}$, respectively, and it follows directly from the definitions that the following Plancherel's identities hold.

Proposition 3 *Let $f_1, f_2 \in L^2(\Omega)$, then $\widehat{f}_1, \widehat{f}_2 \in \ell^2(\mathfrak{L})$, $\widehat{f}_{1*}, \widehat{f}_{2*} \in \ell^2(\mathfrak{L}^*)$ and their inner products take the form*

$$\langle \widehat{f}_1, \widehat{f}_2\rangle_{\ell^2(\mathfrak{L})} = \sum_{\xi\in\mathcal{I}} \widehat{f}_1(\xi)\overline{\widehat{f}_{2*}(\xi)}$$

and

$$\langle \widehat{f}_{1*}, \widehat{f}_{2*}\rangle_{\ell^2(\mathfrak{L}^*)} = \sum_{\xi\in\mathcal{I}} \widehat{f}_{1*}(\xi)\overline{\widehat{f}_2(\xi)},$$

respectively. In particular, we have

$$\overline{\langle \widehat{f}_1, \widehat{f}_2\rangle}_{\ell^2(\mathfrak{L})} = \langle \widehat{f}_{2*}, \widehat{f}_{1*}\rangle_{\ell^2(\mathfrak{L}^*)}$$

and the Parseval identity takes of the form

$$\langle f_1, f_2\rangle_{L^2(\Omega)} = \langle \widehat{f}_1, \widehat{f}_2\rangle_{\ell^2(\mathfrak{L})} = \sum_{\xi\in\mathcal{I}} \widehat{f}_1(\xi)\overline{\widehat{f}_{2*}(\xi)} = \sum_{\xi\in\mathcal{I}} \widehat{f}_{1*}(\xi)\overline{\widehat{f}_2(\xi)}.$$

As a consequence, for any $f \in L^2(\Omega)$ we have $\widehat{f} \in \ell^2(\mathfrak{L})$, $\widehat{f}_* \in \ell^2(\mathfrak{L}^*)$ and

$$\|f\|_{L^2(\Omega)} = \|\widehat{f}\|_{\ell^2(\mathfrak{L})} = \|\widehat{f}_*\|_{\ell^2(\mathfrak{L}^*)}.$$

With these tools at hand, we can define Sobolev spaces associated to the model operator $\mathfrak{L}$ (and $\mathfrak{L}^*$).

Definition 5 For $f \in \mathcal{D}'_{\mathfrak{L}}(\Omega) \cap \mathcal{D}'_{\mathfrak{L}^*}(\Omega)$, and $s \in \mathbb{R}$, we say that $f \in \mathcal{H}^s_{\mathfrak{L}}(\Omega)$ if and only if $\langle\xi\rangle^s \widehat{f}(\xi) \in \ell^2(\mathfrak{L}^*)$. We define the norm on $\mathcal{H}^s_{\mathfrak{L}}(\Omega)$ by

$$\|f\|_{\mathcal{H}^s_{\mathfrak{L}}(\Omega)} := \left(\sum_{\xi\in\mathcal{I}} \langle\xi\rangle^{2s}\widehat{f}(\xi)\overline{\widehat{f}_*(\xi)}\right)^{1/2}.$$

Thus, the Sobolev space $\mathcal{H}^s_{\mathfrak{L}}(\Omega)$ is the space of $\mathfrak{L}$-distributions f for which $\|f\|_{\mathcal{H}^s_{\mathfrak{L}}(\Omega)} < \infty$ and, the same way, we can define the Sobolev space $\mathcal{H}^s_{\mathfrak{L}^*}(\Omega)$ as the space of $\mathfrak{L}^*$-distributions f for which

$$\|f\|_{\mathcal{H}^s_{\mathfrak{L}^*}(\Omega)} := \left(\sum_{\xi\in\mathcal{I}} \langle\xi\rangle^{2s} \widehat{f}_*(\xi)\overline{\widehat{f}(\xi)}\right)^{1/2} < \infty.$$

Notice that, as a consequence of the last proposition, the spaces $\mathcal{H}^s_{\mathfrak{L}}(\Omega)$ and $\mathcal{H}^s_{\mathfrak{L}^*}(\Omega)$ coincide. For every $s \in \mathbb{R}$, the Sobolev space $\mathcal{H}^s_{\mathfrak{L}}(\Omega)$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle_{\mathcal{H}^s_{\mathfrak{L}}(\Omega)} := \sum_{\xi\in\mathcal{I}} \langle\xi\rangle^{2s} \widehat{f}(\xi)\overline{\widehat{g}_*(\xi)}$$

and, similarly, the Sobolev space $\mathcal{H}^s_{\mathfrak{L}^*}(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathcal{H}^s_{\mathfrak{L}^*}(\Omega)} := \sum_{\xi\in\mathcal{I}} \langle\xi\rangle^{2s} \widehat{f}_*(\xi)\overline{\widehat{g}(\xi)},$$

for every $s \in \mathbb{R}$. Moreover, the Sobolev spaces $\mathcal{H}^s_{\mathfrak{L}}(\Omega)$ and $\mathcal{H}^s_{\mathfrak{L}^*}(\Omega)$ are isometrically isomorphic and, in particular, $\mathcal{H}^0_{\mathfrak{L}}(\Omega) = L^2(\Omega)$.

We finish this section mentioning that there is, on the lines of the classical theory, a p-Lebesgue version of the space of Fourier coefficients, i.e. $\ell^p(\mathfrak{L})$-spaces associated to the model operator $\mathfrak{L}$ (and $\mathfrak{L}^*$).

Definition 6 The $\ell^p(\mathfrak{L})$ space associated to the model operator $\mathfrak{L}$ is defined by

$$\ell^p(\mathfrak{L}) := \left\{ a : \mathcal{I} \to \mathbb{C} : \sum_{\xi\in\mathcal{I}} |a(\xi)|^p \left\|u_\xi\right\|^{2-p}_{L^\infty(\Omega)} < \infty \right\} \text{ for } 1 \leq p \leq 2,$$

$$\ell^p(\mathfrak{L}) := \left\{ a : \mathcal{I} \to \mathbb{C} : \sum_{\xi\in\mathcal{I}} |a(\xi)|^p \left\|v_\xi\right\|^{2-p}_{L^\infty(\Omega)} < \infty \right\} \text{ for } 2 \leq p < \infty$$

and

$$\ell^p(\mathfrak{L}) := \left\{ a : \mathcal{I} \to \mathbb{C} : \sup_{\xi\in\mathcal{I}} \left(|a(\xi)| \left\|v_\xi\right\|^{-1}_{L^\infty(\Omega)}\right) < \infty \right\} \text{ for } p = \infty.$$

Spaces $\ell^p(\mathfrak{L}^*)$, for $1 \leq p \leq 2$, $2 \leq p < \infty$ and $p = \infty$, are defined analogously. For more results and applications of these spaces we refer the reader to [5, 10, 11], and references therein.

4.5 Full Symbols and Global Pseudo-Differential Operators Associated to $\mathfrak{L}$ and $\mathfrak{L}^*$

The goal of this section is to present the $\mathfrak{L}$-quantization of the $\mathfrak{L}$-admissible operators induced by our model operator $\mathfrak{L}$ (and, similarly, for $\mathfrak{L}^*$). Recall that a system $\{u_\xi\}_{\xi\in I}$ is called a *WZ-system* if, for all $\xi \in I$, the functions $u_\xi(x)$, $v_\xi(x)$ do not have zeros on the domain and, in addition, there exist a constant $C > 0$ and a non-negative integer N such that, as $|\xi| \to \infty$,

$$\inf_{x\in\Omega} |u_\xi(x)| \geq C\langle\xi\rangle^{-N} \quad \text{and} \quad \inf_{x\in\Omega} |v_\xi(x)| \geq C\langle\xi\rangle^{-N}.$$

From now on we assume that $\{u_\xi\}_{\xi\in I}$ is a WZ-system.

For any linear continuous operator $A : C^\infty_{\mathfrak{L}}(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}}(\Omega)$ there exists a kernel $K_A \in \mathcal{D}'_{\mathfrak{L}}(\Omega \times \Omega)$, called the Schwartz kernel of A, such that for all $f \in C^\infty_{\mathfrak{L}}(\overline{\Omega})$,

$$Af(x) = \int_\Omega K_A(x, y) f(y)\, dy,$$

in the sense of distributions. Thus, making use of the Fourier series formula we have the following expression for the action of A on $C^\infty_{\mathfrak{L}}(\overline{\Omega})$:

$$Af(x) = \sum_{\eta\in I} \widehat{f}(\eta) \int_\Omega K_A(x, y) u_\eta(y) dy.$$

The same follows for any linear continuous operator $A : C^\infty_{\mathfrak{L}^*}(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}^*}(\Omega)$, there exists a kernel $\widetilde{K}_A \in \mathcal{D}'_{\mathfrak{L}^*}(\Omega \times \Omega)$ such that for all $f \in C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$ and, in the sense of distributions, we can write

$$Af(x) = \int_\Omega \tilde{K}_A(x, y) f(y) dy.$$

Using the corresponding Fourier series formula we will have a similar expression for the action of A on $C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$.

Definition 7 Consider a WZ-system $\{u_\xi\}_{\xi\in I}$ and a linear continuous operator $A : C^\infty_{\mathfrak{L}}(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}}(\Omega)$. We say A belongs to the class of $\mathfrak{L}$-*admissible operators* if

$$\sum_{\eta\in I} u_\eta^{-1}(x) u_\eta(z) \int_\Omega K_A(x, y) u_\eta(y) dy \in \mathcal{D}'_{\mathfrak{L}}(\Omega \times \Omega)$$

for all $x, z \in \Omega$. Similarly, a linear continuous operator $A : C^\infty_{\mathfrak{L}^*}(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}^*}(\Omega)$ belongs to the class of $\mathfrak{L}^*$-*admissible operators* if

$$\sum_{\eta\in\mathcal{I}} v_\eta^{-1}(x)v_\eta(z)\int_\Omega K_A(x,y)v_\eta(y)dy$$

belongs to $\mathcal{D}'_{\mathfrak{L}^*}(\Omega\times\Omega)$.

Actually, since $\{u_\xi : \xi\in\mathcal{I}\}$ is a WZ-system, the expression

$$u_\eta^{-1}(x)u_\eta(z)\int_\Omega K_A(x,y)u_\eta(y)dy$$

defines an element in $\mathcal{D}'_{\mathfrak{L}}(\Omega)\otimes\mathcal{S}'(\mathcal{I})$ for any $\mathfrak{L}$-admissible operator A. An analogue definition give rise to $\mathfrak{L}^*$-*admissible operators*.

Consider now a $\mathfrak{L}$-admissible operator A. We will use the $\mathfrak{L}$-Fourier inversion formula to associate to A the analogue of a full or matrix symbol, the $\mathfrak{L}$-symbol of a $\mathfrak{L}$-admissible operator.

Definition 8 The $\mathfrak{L}$-symbol of a linear continuous $\mathfrak{L}$-admissible operator

$$A : C^\infty_{\mathfrak{L}}(\overline{\Omega})\to\mathcal{D}'_{\mathfrak{L}}(\Omega)$$

is defined by

$$\sigma_A(x,\xi) := u_\xi^{-1}(x)\int_\Omega K_A(x,y)u_\xi(y)dy$$

or, equivalently,

$$\sigma_A(x,\xi)=\int_\Omega k_A(x,y)\overline{v_\xi(y)}\mathrm{d}y=\left\langle k_A(x),\bar{v}_\xi\right\rangle$$

where, by the $\mathfrak{L}$-Fourier inversion formula, the convolution kernel can be regained from the symbol as

$$k_A(x,y)=\sum_{\xi\in\mathcal{I}}\sigma_A(x,\xi)u_\xi(y),$$

in the sense of $\mathfrak{L}$-distributions.

The main result in this section is the correspondence between linear continuous $\mathfrak{L}$-admissible operators and $\mathfrak{L}$-symbols, correspondence that we will refer to as $\mathfrak{L}$-*quantization*:

Theorem 3 *Let $A : C^\infty_{\mathfrak{L}}(\overline{\Omega})\to\mathcal{D}'_{\mathfrak{L}}(\Omega)$ be a linear continuous $\mathfrak{L}$-admissible operator with $\mathfrak{L}$-symbol $\sigma_A\in\mathcal{D}'_{\mathfrak{L}}(\Omega)\otimes\mathcal{S}'(\mathcal{I})$. Then the $\mathfrak{L}$-quantization formula*

$$Af(x)=\sum_{\xi\in\mathcal{I}}\widehat{f}(\xi)\sigma_A(x,\xi)u_\xi(x)$$

holds true for all $f \in C^{\infty}_{\mathfrak{L}}(\overline{\Omega})$ *and* $x \in \Omega$. *Then, the* $\mathfrak{L}$*-symbol* σ_A *of* A *can be written as*

$$\sigma_A(x,\xi) = u_{\xi}^{-1}(x)\left(Au_{\xi}\right)(x)$$

for all $x \in \Omega$ *and* $\xi \in \mathcal{I}$.

Consider now a linear continuous $\mathfrak{L}$-admissible operator

$$A : Span\{u_{\xi}\} \subseteq Dom(A) \subseteq L^2(\Omega) \to L^2(\Omega).$$

If there exist a measurable function $\sigma_A : \overline{\Omega} \times \mathcal{I} \to \mathbb{C}$ such that

$$\sigma_A(x,\xi)u_{\xi}(x) = \left(Au_{\xi}\right)(x), \tag{10}$$

then the $\mathfrak{L}$-quantisation formula

$$Af(x) = \sum_{\xi \in \mathcal{I}} \widehat{f}(\xi)\sigma_A(x,\xi)u_{\xi}(x)$$

is valid for all $f \in Span\{u_{\xi}\}$ (and the function $\sigma_A(x,\xi)$ does not need to belong to $\mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(\mathcal{I})$, it is enough that $\sigma_A(\cdot,\xi) \in L^2(\overline{\Omega})$ for each $\xi \in \mathcal{I}$). As a consequence of the above theorem, under condition (10) and under the assumption that the biorthogonal system $\{u_{\xi}\}_{\xi \in \mathcal{I}}$ is a WZ-system, we obtain equivalent formulae for the $\mathfrak{L}$-symbol of A:

Corollary 1 *The* $\mathfrak{L}$*-symbols* $\sigma_A(\cdot,\xi)$ *of* A *can be equivalently obtained from the following expressions in terms of the kernels of* A *and the WZ-system* $\{u_{\xi}\}_{\xi \in \mathcal{I}}$*:*

1. $\sigma_A(x,\xi) = \displaystyle\int_{\Omega} k_A(x,y)\overline{v_{\xi}(y)}\,dy,$
2. $\sigma_A(x,\xi) = u_{\xi}^{-1}(x)\left(Au_{\xi}\right)(x),$
3. $\sigma_A(x,\xi) = u_{\xi}^{-1}(x)\displaystyle\int_{\Omega} K_A(x,y)u_{\xi}(y)dy,$
4. $K_A(x,y) = \displaystyle\sum_{\xi \in \mathcal{I}} u_{\xi}(x)\sigma_A(x,\xi)\overline{v_{\xi}(y)}.$

Remark 3 When $\left\{u_{\xi} : \xi \in \mathcal{I}\right\}$ is not a WZ-system, the $\mathfrak{L}$-symbol σ_A of the operator A is a function on $\overline{\Omega} \times \mathcal{I}$ for which the equality

$$u_{\xi}(x)\sigma_A(x,\xi) = \int_{\Omega} K_A(x,y)u_{\xi}(y)dy$$

holds for all ξ in $\mathcal{I}$ and for $x \in \overline{\Omega}$.

Analogous definitions of classes of $\mathfrak{L}^*$-admissible operators and $\mathfrak{L}^*$-quantization can be given. The $\mathfrak{L}^*$-*symbol* of a linear continuous $\mathfrak{L}^*$-admissible operator $A : C^\infty_{\mathfrak{L}^*}(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}^*}(\Omega)$ is defined by

$$\tau_A(x,\xi) := v_\xi^{-1}(x) \int_\Omega K_A(x,y) v_\xi(y) dy,$$

and, parallel to $\mathfrak{L}$-quantization, the following result gives a complement to the result on $\mathfrak{L}$-quantization given in Theorem 3 before:

Theorem 4 *Let $A : C^\infty_{\mathfrak{L}^*}(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}^*}(\Omega)$ be a linear continuous $\mathfrak{L}^*$-admissible operator with $\mathfrak{L}^*$-symbol $\tau_A \in \mathcal{D}'_{\mathfrak{L}^*}(\Omega) \otimes \mathcal{S}'(\mathcal{I})$. Then the $\mathfrak{L}^*$-quantization*

$$Af(x) = \sum_{\xi \in \mathcal{I}} \widehat{f}(\xi) \tau_A(x,\xi) v_\xi(x)$$

holds for all $f \in C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$ and $x \in \Omega$. Then, the $\mathfrak{L}^*$-symbol τ_A of A can be written as

$$\tau_A(x,\xi) = v_\xi^{-1}(x) \left(A v_\xi\right)(x)$$

for all $x \in \Omega$ and $\xi \in \mathcal{I}$.

Similar assumptions to those used to get Corollary 1, on $\mathfrak{L}^*$-admissible operators $A : Span\{v_\xi\} \subseteq Dom(A) \subseteq L^2(\Omega) \to L^2(\Omega)$, give equivalent formulae for the corresponding $\mathfrak{L}^*$-symbols, which can be found in [10, 11]. We will not give further details on this, instead we will finish this section discussing the notion of $\mathfrak{L}$-Fourier multipliers, i.e. continuous linear operators whose $\mathfrak{L}$-symbols are independent of the variable x.

Definition 9 Let $A : Dom(A) \subseteq L^2(\overline{\Omega}) \to L^2(\overline{\Omega})$ be a continuous linear operator. We call A a $\mathfrak{L}$-Fourier multiplier if, for any $f \in Dom(A)$,

$$\mathcal{F}_{\mathfrak{L}}(Af)(\xi) = \sigma(\xi) \mathcal{F}_{\mathfrak{L}}(f)(\xi),$$

for some $\sigma : \mathcal{I} \to \mathbb{C}$. Similarly, if $B : Dom(A) \subseteq L^2(\overline{\Omega}) \to L^2(\overline{\Omega})$ is a continuous linear operator, we call B a $\mathfrak{L}^*$-*Fourier multiplier* if, for any $f \in Dom(B)$, it satisfies

$$\mathcal{F}_{\mathfrak{L}^*}(Bf)(\xi) = \tau(\xi) \mathcal{F}_{\mathfrak{L}^*}(f)(\xi),$$

for some $\tau : \mathcal{I} \to \mathbb{C}$.

The relation between the symbols of a $\mathfrak{L}$-Fourier multiplier A and its adjoint A^* is given in the next proposition.

Proposition 4 *The continuous linear operator A is a $\mathfrak{L}$-Fourier multiplier, with $\mathfrak{L}$-symbol $\sigma(\xi)$, if and only if A^* is a $\mathfrak{L}^*$-Fourier multiplier with $\mathfrak{L}^*$-symbol $\overline{\sigma(\xi)}$.*

4.6 Difference Operators and a Global Symbolic Calculus for Pseudo-Differential Operators Associated to Boundary Value Problems

Let us now come back to the difference operators mentioned before, in the case of matrix symbols associated to continuous linear operators on compact Lie groups, which can be defined in the context of boundary value problems as the ones considered in this chapter. The first step is the definition of $\mathfrak{L}$-strongly admissible (and $\mathfrak{L}^*$-strongly admissible) functions. We will make use of the following notation

$$C_b^\infty(\Omega \times \Omega) = C^\infty(\Omega \times \Omega) \cap C^\infty(\overline{\Omega} \times \overline{\Omega}).$$

Definition 10 Consider a family of smooth functions $q_j \in C_b^\infty(\Omega \times \Omega)$, for $j = 1, \ldots, l$. The collection of q_j's are called $\mathfrak{L}$-*strongly admissible* if the following properties holds:

1. For every $x \in \Omega$, the multiplication by $q_j(x, \cdot)$ is a continuous linear mapping on $C^\infty_{\mathfrak{L}^*}(\overline{\Omega})$, for all $j = 1, \ldots, \ell$;
2. $q_j(x, x) = 0$ for all $j = 1, \ldots, \ell$;
3. rank $\big(\nabla_y q_1(x, y), \ldots, ; \nabla_y q_l(x, y)\big)\big|_{y=x} = \dim(\Omega)$ at every point $x \in \Omega$;
4. the only set in which all of the q_j's vanish is the diagonal in $\Omega \times \Omega$, i.e.

$$\bigcap_{j=1}^{l} \big\{(x, y) \in \Omega \times \Omega : q_j(x, y) = 0\big\} = \{(x, x) : x \in \Omega\}.$$

A collection of q_j's satisfying the properties above provides a generalization of the notion of a strongly admissible collection of functions, for difference operators in the context of compact Lie groups, introduced in [13]. Here we use the standard conventions for powers of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_l)$, i.e. $q^\alpha(x, y) := q_1^{\alpha_1}(x, y) \cdots q_l^{\alpha_l}(x, y)$. Analogously, there is a notion of $\mathfrak{L}^*$-strongly admissible functions which we omit.

We can find a Taylor expansion formula for smooth functions, with respect to a $\mathfrak{L}$-strongly admissible family of functions $q_j \in C_b^\infty(\Omega \times \Omega)$, for $j = 1, \ldots, l$, which follows from the Taylor series of the functions $q^\alpha(e, \cdot)$.

Proposition 5 *Any smooth function $g \in C^\infty(\Omega)$ can be approximated by Taylor polynomial type expansions around $e \in \Omega$, i.e.*

$$\begin{aligned} g(x) &= \sum_{|\alpha|<N} \frac{1}{\alpha!} D_x^{(\alpha)} g(x)\big|_{x=e} q^\alpha(e, x) + \sum_{|\alpha|=N} \frac{1}{\alpha!} q^\alpha(e, x) g_N(x) \\ &\sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} D_x^{(\alpha)} g(x)\big|_{x=e} q^\alpha(e, x) \end{aligned}$$

in a neighborhood of $e \in \Omega$, where $g_N \in C^\infty(\Omega)$ and $D_x^{(\alpha)} g(x)\big|_{x=e}$ can be calculated from a recurrent relation. Indeed, $D_x^{(0,\cdots,0)} := I$ and, for $\alpha \in \mathbb{N}_0^\ell$, we have

$$\partial_x^\beta g(x)\big|_{x=e} = \sum_{|\alpha|\leq|\beta|} \frac{1}{\alpha!} \left[\partial_x^\beta q^\alpha(e,x)\right]\big|_{x=e} D_x^{(\alpha)} g(x)\big|_{x=e},$$

where $\beta = (\beta_1, \cdots, \beta_n)$ and $\partial_x^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$.

Similarly, for the adjoint problem (i.e. the associated to $\mathfrak{L}^*$-strongly admissible families), any smooth function $g \in C^\infty(\Omega)$ can be approximated by Taylor polynomial type expansions given by

$$\begin{aligned} g(x) &= \sum_{|\alpha|<N} \frac{1}{\alpha!} \tilde{D}_x^{(\alpha)} g(x)\big|_{x=e} \tilde{q}^\alpha(e,x) + \sum_{|\alpha|=N} \frac{1}{\alpha!} \tilde{q}^\alpha(e,x) g_N(x) \\ &\sim \sum_{\alpha\geq 0} \frac{1}{\alpha!} \tilde{D}_x^{(\alpha)} g(x)\big|_{x=e} \tilde{q}^\alpha(e,x) \end{aligned}$$

in a neighborhood of $e \in \Omega$, with $g_N \in C^\infty(\Omega)$ and $\tilde{D}_x^{(\alpha)} g(x)\big|_{x=e}$ calculated from a similar recurrent relation. Notice that the operators $D^{(\alpha)}$ and $\tilde{D}^{(\alpha)}$ are standard differential operators of order $|\alpha|$, and the operator ∂_x^β can be written as a linear combinations of the operators $D^{(\alpha)}$ or $\tilde{D}^{(\alpha)}$, with smooth bounded coefficients. Also, note that difference operators applied to symbols will depend on a point x since, in the generality we are working with, the problem may lack of invariance or symmetry. Now we are ready to give the definition of difference operators we are going to use to state the main results of this section.

Definition 11 Let $A : C_{\mathfrak{L}}^\infty(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}}(\Omega)$ be a linear continuous $\mathfrak{L}$-admissible operator with $\mathfrak{L}$-symbol $\sigma_A \in \mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(I)$ and Schwartz kernel $K_A(x,y) \in \mathcal{D}'_{\mathfrak{L}}(\Omega \times \Omega)$. Then, the difference operator

$$\Delta_q^\alpha : \mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(I) \to \mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(I),$$

acts on $\mathfrak{L}$-symbols σ_A by

$$\Delta_q^\alpha \sigma_A(x,\xi) = u_\xi^{-1}(x) \int_\Omega q^\alpha(x,y) K_A(x,y) u_\xi(y) dy.$$

Analogously, for a $\mathfrak{L}^*$-admissible operator $A : C_{\mathfrak{L}^*}^\infty(\overline{\Omega}) \to \mathcal{D}'_{\mathfrak{L}^*}(\Omega)$ with $\mathfrak{L}^*$-symbol $\tau_A \in \mathcal{D}'_{\mathfrak{L}^*}(\Omega) \otimes \mathcal{S}'(I)$ and Schwartz kernel $\tilde{K}_A(x,y) \in \mathcal{D}'_{\mathfrak{L}^*}(\Omega \times \Omega)$, the difference operator

$$\tilde{\Delta}_q^\alpha : \mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(I) \to \mathcal{D}'_{\mathfrak{L}}(\Omega) \otimes \mathcal{S}'(I),$$

acts on the $\mathfrak{L}^*$-symbol τ_A by

$$\tilde{\Delta}_q^{\alpha}\tau_A(x,\xi) = v_{\xi}^{-1}(x)\int_{\Omega}\tilde{q}^{\alpha}(x,y)\tilde{K}_A(x,y)v_{\xi}(y)dy.$$

Using the difference operators and the derivatives $D^{(\alpha)}$, we come to the main goal of the construction, the symbol classes $S^m_{\rho,\delta}(\Omega\times\mathcal{I})$, for $m\in\mathbb{R}$ and $0\leq\rho,\delta\leq 1$, associated to the boundary value problem $(\Omega,\mathfrak{L})$.

Definition 12 The $\mathfrak{L}$-symbol class $S^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$ consist of symbols $\sigma(x,\xi)$ in $C^{\infty}_{\mathfrak{L}}\left(\overline{\Omega}\right)$, for all $\xi\in\mathcal{I}$, satisfying

$$\left|\Delta^{\alpha}_{(x)}D^{(\beta)}_x\sigma(x,\xi)\right|\leq C_{\sigma,\alpha,\beta,m}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all $\alpha,\beta\geq 0$ and $x\in\overline{\Omega}$. Similarly, the $\mathfrak{L}^*$-symbol class $\tilde{S}^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$ consist of symbols $\tau(x,\xi)$ in $C^{\infty}_{\mathfrak{L}^*}\left(\overline{\Omega}\right)$, for all $\xi\in\mathcal{I}$, satisfying

$$\left|\tilde{\Delta}^{\alpha}_{(x)}\tilde{D}^{(\beta)}_x\tau(x,\xi)\right|\leq C_{\sigma,\alpha,\beta,m}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all $\alpha,\beta\geq 0$ and $x\in\overline{\Omega}$.

The symbol classes $S^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$ and $\tilde{S}^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$ give rise to global $\mathfrak{L}$-pseudo-differential operators (and global $\mathfrak{L}^*$-pseudo-differential operators), of order $m\in\mathbb{R}$, using the natural generalization of the Fourier transform (associated to $\mathfrak{L}$ and $\mathfrak{L}^*$) given before, and formula (5). The reader can verify the perfect match between this result and the one corresponding to local symbols, as discussed before, corresponding to classical symbol classes (of Hörmander and others) using usual derivatives with respect to the dual (continuous variable) ξ instead of difference operators.

Definition 13 Let $\sigma:\overline{\Omega}\times\mathcal{I}\to\mathbb{C}$ be a measurable function such that $\sigma(\cdot,\xi)\in L^2(\Omega)$ for all $\xi\in\mathcal{I}$. Then the $\mathfrak{L}$-*pseudo-differential operator* corresponding to σ is the linear operator given by

$$T_{\sigma}f(x) := \sum_{\xi\in\mathcal{I}}\sigma(x,\xi)\hat{f}(\xi)u_{\xi}(x), \tag{11}$$

for all $f\in Span\{u_{\xi}\}$. This $\sigma(x,\xi)$ is called the $\mathfrak{L}$-symbol of the pseudo-differential operator T_{σ}. Analogously, for a measurable function $\tau:\overline{\Omega}\times\mathcal{I}\to\mathbb{C}$ such that $\tau(\cdot,\xi)\in L^2(\Omega)$ for all $\xi\in\mathcal{I}$, the $\mathfrak{L}^*$-*pseudo-differential operator* corresponding to τ is given by

$$T_{\tau}f(x) := \sum_{\xi\in\mathcal{I}}\tau(x,\xi)\hat{f}_*(\xi)v_{\xi}(x),$$

for all $f\in Span\{v_{\xi}\}$.

The pseudo-differential calculi obtained from these classes of symbols, which generalize that defined with matrix symbols in the case of compact Lie groups in Sect. 3, share the features of the calculus of pseudo-differential operators defined by classical symbol classes (see e.g. Hörmander [7]). Following the usual notations from the classical theory, we will denote $S^m_{1,0}(\overline{\Omega}\times\mathcal{I})$ by $S^m(\overline{\Omega}\times\mathcal{I})$ and $\tilde{S}^m_{1,0}(\overline{\Omega}\times\mathcal{I})$ by $\tilde{S}^m(\overline{\Omega}\times\mathcal{I})$, respectively. Also,

$$S^\infty_{\rho,\delta}(\overline{\Omega}\times\mathcal{I}):=\bigcap_{m\in\mathbb{R}}S^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})\quad\text{and}\quad S^\infty(\overline{\Omega}\times\mathcal{I}):=\bigcap_{m\in\mathbb{R}}S^m(\overline{\Omega}\times\mathcal{I})$$

(which have the corresponding analogue for $\mathfrak{L}^*$-classes) give the corresponding notion of smoothing symbols, in terms of which many important asymptotic formulae can be established. As expected, a countable family of seminorms can be defined on these spaces of symbols, and they define a Fréchet topology on symbols which will a topology also to the space of operators [10]. Moreover, a composition formula in terms of difference operators and symbols, as well as a formula for the adjoint operator from the symbol of the operator can be given explicitly.

Theorem 5 *1. Let $m_1,m_2\in\mathbb{R}$ and $\rho>\delta\geq 0$, and consider two operators $A,B:C^\infty_{\mathfrak{L}}(\overline{\Omega})\to C^\infty_{\mathfrak{L}}(\overline{\Omega})$, with $\mathfrak{L}$-symbols σ_A and σ_B satisfying, respectively, $\left|\Delta^\alpha_{(x)}\sigma_A(x,\xi)\right|\leq C_\alpha\langle\xi\rangle^{m_1-\rho|\alpha|}$ and $\left|D^{(\beta)}_x\sigma_B(x,\xi)\right|\leq C_\beta\langle\xi\rangle^{m_2+\delta|\beta|}$, for all $\alpha,\beta\geq 0$, uniformly in $x\in\overline{\Omega}$ and $\xi\in\mathcal{I}$. Then, the composition operator has a $\mathfrak{L}$-symbol satisfying, for every $N\in\mathbb{N}$,*

$$\left|\sigma_{AB}(x,\xi)-\sum_{\alpha\geq 0}\frac{1}{\alpha!}\left(\Delta^\alpha_{(x)}\sigma_A(x,\xi)\right)D^{(\alpha)}_x\sigma_B(x,\xi)\right|\leq C_N\langle\xi\rangle^{m_1+m_2-(\rho-\delta)N},$$

i.e. the following asymptotic expansion

$$\sigma_{AB}(x,\xi)\sim\sum_{\alpha\geq 0}\frac{1}{\alpha!}\left(\Delta^\alpha_{(x)}\sigma_A(x,\xi)\right)D^{(\alpha)}_x\sigma_B(x,\xi).\tag{12}$$

2. *Let $m\in\mathbb{R}$ and $0\leq\rho,\delta\leq 1$, and consider $\sigma\in S^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$. Assume that the conjugate symbol class $\tilde{S}^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$ is defined with strongly admissible functions $\tilde{q}_j(x,y):=\overline{q_j(x,y)}$ which are $\mathfrak{L}^*$-admissible. Then, the operator T^*_σ (i.e. the adjoint of T_σ) has $\mathfrak{L}^*$-symbol $\tau:\overline{\Omega}\times\mathcal{I}\to\mathbb{C}$ in $\tilde{S}^m_{\rho,\delta}(\overline{\Omega}\times\mathcal{I})$ with asymptotic expansion*

$$\tau(x,\xi)\sim\sum_{\alpha}\frac{1}{\alpha!}\tilde{\Delta}^\alpha_{(x)}D^{(\alpha)}_x\overline{\sigma(x,\xi)}.\tag{13}$$

This theorem is just the starting point for the study of the algebra of global pseudo-differential operators associated to the boundary value problem $(\Omega,\mathfrak{L})$. Properties

of Schwartz kernels of pseudo-differential operators with symbols in the introduced Hörmander-type classes defined before, ellipticity of operators defined by full symbols in classes generated by $\mathfrak{L}$ and $\mathfrak{L}^*$, as well as parametrix and a priori estimates for global pseudo-differential operators in, for example, Sobolev spaces, Schatten classes and nuclearity associated to $\mathfrak{L}$, have been studied in [5, 10, 11] and subsequent papers. It is worth to mention that the condition imposed on the system $\{u_\xi\}_{\xi\in\mathcal{I}}$, at the beginning of Sect. 4.5, namely that the system of eigenfunctions is without zeros, can be dropped out, and the same for the boundedness condition on the domain Ω. The reader can visit the website https://ruzhansky.org/ (open access sources) for examples, new results and an extensive list of references in this subject.

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Mathematical Foundations of Topological Matter

Manuel Asorey and Yisely Martínez

Abstract In these lecture notes we introduce the basic geometric and topological elements for the foundations of topological matter. This new field of condensed matter which ranges from topological quantum phases to topological insulators and topological semimetals has undergone an impressive development in the last years, motivated by its potential applications to spintronics and quantum computation. We outline the main mathematical foundations of the new field.

1 Introduction

Topological matter is a new branch of physics which describes quantum states of matter that are characterized by topological invariants which provides them a robust topological protection crucial for interesting applications. This includes topological quantum phases of matter, topological insulators and topological semimetals [1–3].

One interesting characteristic of some of the new topological materials is that its bulk is insulating while its edge is metallic [3–5]. This striking property is made possible by the appearance of conducting edge states. The quantized conducting properties of these materials are usually characterized by topological indices that either depend on the bulk observables of the material [6–8] or on a number of edge states with special characteristics [9, 10]. The amazing coincidence of the two ways of computing the quanta of conductivity is attributed to a mysterious bulk-edge correspondence. The appealing image of such a correspondence has been in some cases related to the AdS/CFT duality arising from string theory which provides the basis for holographic principles.

However, the bulk-edge correspondence has never been rigorously proven. In these notes we will show that the existence of the bulk-edge duality is highly dependent

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on the physical properties of the material and its boundary conditions [11, 12]. One of the goals of the present lectures is to try to clarify this issue.

From a mathematical viewpoint the difference between insulators and conductors is that the low energy spectrum of the Hamiltonian in conductors is a single connected domain of the real line whereas in the insulators is a multiple connected domain made of different intervals (bands) separated by void energy gaps. In the later case the Fermi level controls the conducting-insulating transition of the corresponding material. When the Fermi level is located in the spectral gaps the behavior is insulating and otherwise is conducting.

2 Quantum Phase Transitions and Level Crossing

A characteristic of topological matter is the existence of new phases of matter where the corresponding order parameters are given by highly non-local operators [13] which in fact are sensible to some underpinning topology of the quantum system. Another characteristic of some of those systems is that they undergo new types of phase transitions known as quantum phase transitions, associated to the existence of a degenerated quantum vacuum (see e.g. Ref. [1] for a review). Generically, the degeneracy of the ground state is highly dependent on the topology of the configuration space.

However, for quantum systems without internal degrees of freedom evolving under real smooth potentials in $\mathbb{R}^n$, the quantum vacuum is non-degenerate. In this scenario quantum phase transitions never occur.

The proof of this property follows from the time reversal invariance of real Hamiltonians of the form [14]

$$H = \frac{1}{2}\Delta + V(x), \tag{1}$$

defined by the Laplace–Beltrami operator $\Delta = d^*d$ and a non-singular potential $V(x)$. Unitarity of quantum evolution requires the potential $V(x)$ to be real, $V(x) = V(x)^*$, to guarantee the hermicity of H. In this case the system is invariant under time reversal,

$$U(T)\psi(x,t) = \psi(x,-t)^*, \qquad E_0 > 0$$

because $[U(T), H] = 0$. This symmetry implies that for any energy level there is a basis of real stationary states, $\psi_n(x) = \psi_n(x)^*$. Indeed, if $\psi_n(x)$ is an eigenstate of H with energy E_n, the state $\psi_n(x)^*$ is also an eigenstate with the same energy. If ψ_n is not real, $\psi_n^* \neq \psi_n$, the states $\psi_+ = \psi_n^* + \psi_n$ and $\psi_- = i(\psi_n^* - \psi_n)$will have the same energy E_n and will be real, irrespectively of the degeneracy or not of the energy level.

If V is bounded below (let us assume that $V > 0$) and has a non-trivial discrete spectrum there is a ground state ψ_0 whose energy attains the minimum $E_0 > 0$ of

the energy spectrum. Because V has no singularities on $\mathbb{R}^n$ it is trivial to see that ψ_0 cannot vanish for any point x of the configuration space. Indeed, ψ_0 satisfies the stationary equation

$$H\psi_0 = E_0\psi_0.$$

If the set of nodal points of ψ_0, $\mathcal{N}_0 = \{x \in M;\, \psi_0(x) = 0\}$, is non-trivial, the positive real function $\psi_1 = |\psi_0|$ is smooth everywhere except perhaps at the points of $\mathcal{N}_0$. The expectation value of the Hamiltonian on the state ψ_1 is again E_0 because the delta function singularity of $\Delta\psi_1$ at $\mathcal{N}_0$ is cancelled by the vanishing of ψ_1 at that point,

$$E_0 = \langle\psi_0|H\psi_0\rangle = \langle\psi_1|H\psi_1\rangle.$$

Since E_0 is the lowest eigenvalue of H this means that ψ_1 is also an eigenstate of H with the same energy that ψ_0. Now, elliptic regularity implies that any eigenstate of H must be smooth. Thus, if $\psi_1 \neq \psi_0$ the state ψ_1 cannot be an eigenstate because then its differentials are discontinuous at $\mathcal{N}_0$. The contradiction, being motivated by the assumption of existence of nodes, disappears if $\psi(x) \neq 0$ for any $x \in M$.

If $\psi_1 = \psi_0$ we can choose the wave function of the ground state to be definite non-negative $\psi_0(x) \geq 0$ for any $\mathbb{R}^n$, vanishing only at $\mathcal{N}_0$. In that case the previous argument does not work. The ground state wave function ψ_0 attaints a minimal value at the points of $\mathcal{N}_0$, i.e. $d\psi_0(x) = \psi_0(x) = 0$ for any point $x \in \mathcal{N}_0$. Moreover,

$$0 = E_0\psi_0(x) = H\psi_0(x) = \frac{1}{2}\Delta\psi_0(x) = \Delta^n\psi_0(x), \quad \text{for any } x \in \mathcal{N}_0 \text{ and } n \in \mathbb{N} \tag{2}$$

and, due to the regularity properties of ψ_0, this implies that $\mathcal{N}_0$ is either empty $\mathcal{N}_0 = \emptyset$ or the full configuration space $\mathcal{N}_0 = \mathbb{R}^n$. This shows that the ground state wave functional cannot have nodal points $\psi(x) \neq 0$. The situation changes if the wave function has internal degrees of freedom. The main reason for this different behaviour is topological.

The same argument leads to the proof of vacuum uniqueness. If the vacuum were degenerate, there would exist another ground state $\psi_1 \neq \lambda\psi_0$. Then, the ground state defined by $\chi(x) = \psi_0(x_*)\psi_1(x) - \psi_1(x_*)\psi_0(x)$ will vanish for $x = x_*$, which cannot occur by the previous argument.

Both results essentially rely on the real, local and smooth characteristics of the potential V. Exceptions for this archetypical infrared behaviour of quantum systems can arise by the introduction either of internal degrees of freedom (e.g. spin), singular or non-local potentials, or complex interactions.

Complex interactions are physically generated by the presence of magnetic fields. The interaction with the magnetic gauge field potential A through the minimal coupling gauge principle leads to a Hamiltonian

$$H_A = \frac{1}{2}\Delta_A + V(x), \qquad \Delta_A = d_A^* d_A, \quad d_A = d - ieA \tag{3}$$

which is not invariant under time reversal, $U(T)H_A U(T) = H_{-A}$. The eigenstates are not necessarily real functions and the rest of the argument leading to the absence of nodes and uniqueness of the vacuum state fails.

There are some interesting cases of quantum phase transitions where the degeneracy of the ground state is highly dependent on the topology of the configuration space. These systems have interest for different physical applications like the quantum Hall effect [15] and quantum circuits which are useful resources in quantum information theory.

3 Topology of Metallic Bands and Bloch–Floquet Theorems

The quantum dynamics of electrons moving on the periodic potential of a perfect cubic crystal has a discrete translation symmetry group $\mathbb{Z}^d$. The periodicity of the system reduces its effective space to a single cell with the topology of a torus

$$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d,$$

known as Brillouin zone. In a cubic crystal the translation symmetry group $\mathbb{Z}^d$ is generated by the lattice of spacial translations $T_j = e^{iap_j}$, $j = 1, 2, \ldots, d$ where a is the crystal cell period and $p_j = -i\,\partial_j$ the momentum operator.

If we neglect the internal degrees of freedom of the electron, the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ can be decomposed as a direct sum of irreducible representations of $\mathbb{Z}^d$

$$\mathcal{H} = \bigoplus_{\epsilon\in\mathbb{T}^d} \mathcal{H}_\epsilon, \tag{4}$$

where $\epsilon \in \mathbb{T}^d$ is the label of the irreducible representation of $\mathbb{Z}^d$, i.e.

$$T_j\psi_\epsilon(x) = e^{-ia\epsilon_j}\psi_\epsilon(x) \qquad 0 \le \epsilon_j \le \frac{2\pi}{a}, \tag{5}$$

which defines de Brillouin zone of the crystal. Now, since the periodic Hamiltonian

$$H = -\tfrac{1}{2}\Delta + V(x), \qquad V(x+a) = V(x) \tag{6}$$

of the system commutes with the translations symmetry group $\mathbb{Z}^d$, it can be decomposed in diagonal blocks labeled by the irreducible representations of $\mathbb{Z}^d$.

$$H = \bigoplus_{\epsilon\in\mathbb{T}^d} H_\epsilon, \tag{7}$$

where each Hamiltonian is defined on a Hilbert space $\mathcal{H}_\epsilon$ of the Bloch–Floquet decomposition theorem, i.e., H can be considered as an operator acting on a bundle over $\mathbb{T}^d$ with fibers the Hilbert spaces $\mathcal{H}_\epsilon$ defined by the states satisfying the boundary conditions

$$\psi_\epsilon(x_j + a) = e^{-ia\epsilon_j}\psi_\epsilon(x). \tag{8}$$

In mathematical terms the physical problem of the dynamics of the electron on an Euclidean space $\mathbb{R}^d$ has been reduced to a family of dynamics on the torus $\mathbb{T}^d$ governed by the selfadjoint extensions of the periodic Hamiltonian (6) with pseudoperiodic boundary conditions (8). This technique is known as compactification.

The spectral problem of Hamiltonian (6) has been reduced to a family of spectral problems of H_ϵ. In particular, the continuous spectrum of H has been reduced to a family of discrete eigenvalue problems.

Let us consider the case of one-dimensional systems $d = 1$ in absence of external forces, i.e. $V = 0$ and

$$H = -\tfrac{1}{2}\partial_x^2. \tag{9}$$

In that case the spectrum is a continuous half real line $\mathbb{R}^+$. There are no gaps inside the spectrum. The system is invariant under any kind of translation. Thus any period a is valid. Let us consider $a = 2\pi$. The family of Hamiltonians of the Bloch–Floquet decomposition do have a non-trivial discrete spectrum

$$E_n^\epsilon = \tfrac{1}{2}(n - \epsilon)^2, \tag{10}$$

associated to normalizable states

$$\psi_n^\epsilon(x) = e^{i(n-\epsilon)x}, \tag{11}$$

on $S^1 = \mathbb{T} = \{-\dfrac{1}{2} \leq \epsilon \leq \dfrac{1}{2}\}$. In fact, by means of a non-periodic gauge transformation

$$\psi^\epsilon(x) \to \psi'^\epsilon(x) = e^{i\epsilon x}\psi^\epsilon(x), \tag{12}$$

the system is reduced to a family of Hamiltonians

$$H'_\epsilon = e^{i\epsilon x} H_\epsilon e^{-i\epsilon x} = -\tfrac{1}{2}\left(\partial_x - i\epsilon\right)^2. \tag{13}$$

with a common domain, the space of periodic boundary conditions

$$\psi'_\epsilon(x + 2\pi) = \psi'_\epsilon(x). \tag{14}$$

Notice that the level crossing occurs for values of Bloch wave parameter $\epsilon = \pm\dfrac{1}{2}$ which are invariant under time reversal symmetry on the torus $\mathbb{T}$ (Fig. 1).

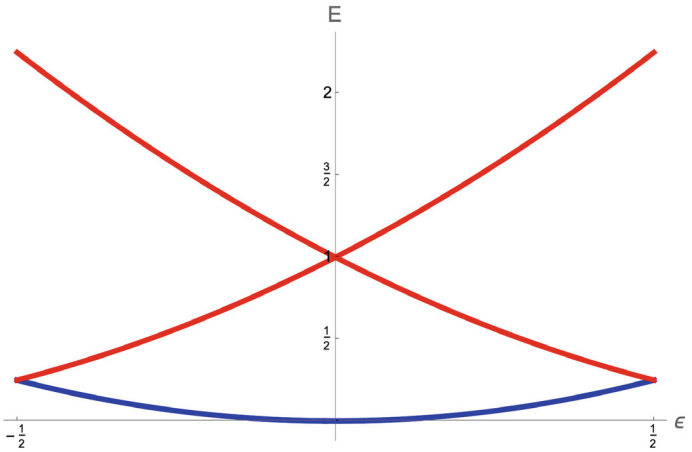

Fig. 1 Low energy levels of the planar rotor for $0 \leq \varepsilon \leq 1$. The blue curve at the bottom represent the ground state energies. Notice the crossing of the ground state energy level with the first excited state at the quantum transition points $\varepsilon = \pm\frac{1}{2}$, where the ground state becomes degenerate (Kramers points)

Although the Hamiltonian (9) is invariant under time reversal,

$$U(T)\psi(x,t) = \psi(x,-t)^*,$$

because $[U(T), H] = 0$, the individual Bloch–Floquet Hamiltonians (13) are not. In fact, we have

$$U(T)H'_\epsilon U(T) = e^{-ix} H'_{1-\epsilon} e^{ix},$$

which means that H'_ϵ and $H'_{1-\epsilon}$ are equivalent up to a unitary gauge transformation. This asymmetry is due to the fact that ϵ plays the role of a magnetic field which is not time reversal invariant. Notice that Kramers points $\epsilon = \pm\frac{1}{2}$ are fixed points under a time reversal transformation, and

$$U(T)H'_{1/2}U(T) = e^{-ix} H'_{1/2} e^{ix}.$$

The existence of level crossing is due to the same reason. For Hamiltonians of systems without internal degrees of freedom that are strictly time reversal invariants the ground state is unique. But in the case of $\epsilon = 1/2$ the Bloch–Floquet Hamiltonians (13) are invariant up to a gauge transformation, which means that the energy levels are degenerate.

Due to these crossings there is no gap in the spectrum of the Hamiltonian H on the line $\mathbb{R}$ and the systems is a perfect conductor. However, the level crossing is not stable under perturbations. In particular, if we consider the perturbation

$$H = -\tfrac{1}{2}\partial_x^2 + \lambda \sin^2 \tfrac{x}{2}. \tag{15}$$

Notice that the perturbation preserves the invariance under space translations $T\psi(x) = \psi(x + 2\pi)$ and time reversal. However, in the spectra of the corresponding Bloch–Floquet Hamiltonians

$$H'_\epsilon = -\tfrac{1}{2}(\partial_x - i\epsilon)^2 + \lambda \sin^2 x \tag{16}$$

the lowest energy levels do not cross each other (see Fig. 2). In this case the time reversal symmetry does not guarantee the presence of level degeneracy at time reversal invariant points (Fig. 3). However, in the presence of spin and anti-unitary representations of time reversal symmetry the stability of such crossings is guaranteed by the Kramers theorem [16].

The same phenomenon occurs in $d = 2$ and $d = 3$ dimensions. In the two dimensional case we get a Brillouin zone which is also a 2D torus $\widehat{\mathbb{T}}^2$, where the corresponding free Bloch–Floquet Hamiltonians are

$$H_\epsilon = -\frac{1}{2}(\partial_1 - i\epsilon_1)^2 - \frac{1}{2}(\partial_2 - i\epsilon_2)^2. \tag{17}$$

The spectrum is the sum of the spatial components and again for each fixed values of the Bloch parameters $(\varepsilon_1, \varepsilon_2) \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] = \widehat{\mathbb{T}}^2$ is discrete. But all together fill the positive energy halfline, due to fact that there are many crossings of the discrete energy levels as far as ϵ varies on the dual torus $\widehat{\mathbb{T}}^2$.

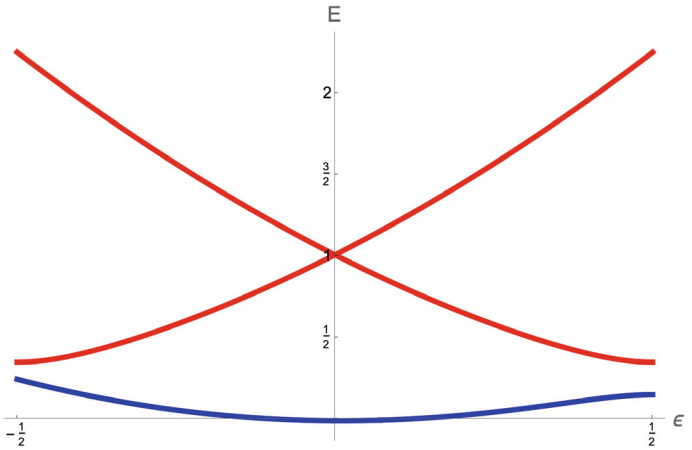

Fig. 2 Low energy levels of the Hamiltonian (16) with $\lambda = 0.1$. The blue curve at the bottom represents the ground state energies. Notice the total absence of level crossing between the two lowest energy levels

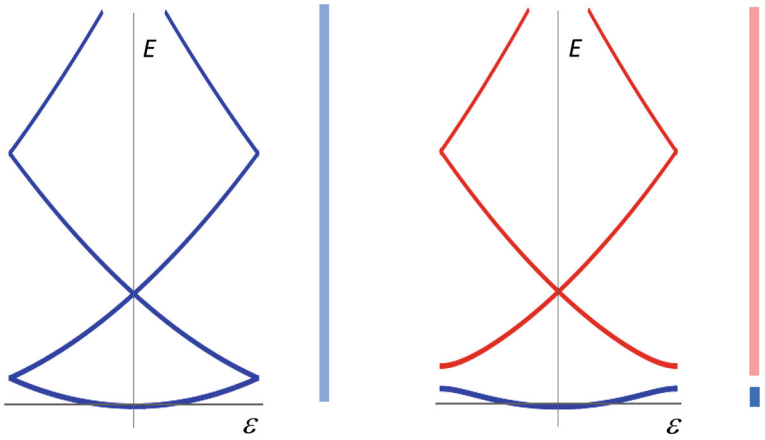

Fig. 3 Low energy levels of the Hamiltonian (16) with $\lambda = 0$ and $\lambda = 0.1$. In the first case we observe a perfect matching of the different energy levels without gaps thanks to the existence of level crossings. In the second case the appearance of gaps is due to the level repulsion induced by the perturbation. In the first case we have a conducting metal, whereas in the second case for values of Fermi energies inside the gap we have an insulator

If we introduce a periodic interaction generated by the crystal nuclei we have a perturbed Hamiltonian

$$H_\epsilon^\lambda = -\frac{1}{2}\,(\partial_1 - i\epsilon_1)^2 - \frac{1}{2}\,(\partial_2 - i\epsilon_2)^2 + \lambda(\sin^2 x_1 + \sin^2 x_2). \tag{18}$$

The energy spectrum is the sum of the two spatial components

$$E^\lambda_{n_1,n_2} = E^\lambda_{n_1} + E^\lambda_{n_2}, \tag{19}$$

and in this case the lowest energy bands do not cross each other and the theory presents a mass gap which gives rise to an insulator phase.

4 Edge States

If instead of having an infinite cubic crystal plate which leads via Bloch–Floquet theorem to a system living in a 2D torus we have an infinite strip of transversal size $2L$. In this case the Bloch–Floquet theorem leads to a family of systems living in on a cylinder of size $2L$ limited by two circles $S^1_\pm$ (Fig. 4). The most general boundary conditions which preserve the Hermiticity of the Hamiltonian (17) are given by [17]

Fig. 4 Brillouin zone of a free electron moving on a strip plate of infinite longitudinal size. The radius of the cylinder depends on the periodic size of the crystal

$$\begin{pmatrix} \phi_1(L) - i\partial_2\phi(L) \\ \phi_1(-L) - i\partial_2\phi(-L) \end{pmatrix} = e^{i\alpha}(\cos\beta\mathbb{I} + i\sin\beta\,\boldsymbol{\sigma}\cdot\mathbf{n}) \begin{pmatrix} \phi_1(L) + i\partial_2\phi(L) \\ \phi_1(-L) + i\partial_2\phi(-L) \end{pmatrix} \tag{20}$$

where $\mathbf{n}$ is an arbitrary unit vector. The edge states are given by wave functions of the form

$$Ae^{\kappa x_2} + Be^{-\kappa x_2} \tag{21}$$

that satisfy the boundary conditions (20). The values of κ are then given by the zeros of the spectral function [18]

$$\begin{aligned} h(i\kappa) = 2ie^{i\alpha}\big[&-i\sinh(\kappa L)\left((1+\kappa^2)\cos\beta + (1-\kappa^2)\cos\alpha\right) \\ &- 2i\kappa\sin\alpha\cosh(\kappa L) - 2i\kappa n_1\sin\beta\big]. \end{aligned} \tag{22}$$

In the large L limit the edge state κ parameter is given by the solutions of

$$(1+\kappa^2)\cos\beta + (1-\kappa^2)\cos\alpha + 2\kappa\sin\alpha = 0, \tag{23}$$

i.e.

$$\kappa = -\frac{\sin\alpha \pm \sin\beta}{\sin\alpha + \sin\beta}. \tag{24}$$

However, the physical requirement that $0 < \kappa < \infty$ implies that for many boundary conditions there are not edge states. For instance for Dirichlet ($\alpha = \pi, \beta = 0$), Newmann ($\alpha = 0, \beta = 0$) and periodic boundary conditions ($\alpha = \frac{3\pi}{2}, \beta = \frac{1}{2}$) there is no edge state.

In the cases where there are edge states they can have negative energies

$$E_k = (n-\epsilon)^2 - \kappa^2. \tag{25}$$

5 Dirac Operator

Another paradigmatic two-dimensional system that is very interesting from the mathematical viewpoint is the Dirac system with Hamiltonian a Dirac operator

$$H_D = -i\sigma_1\partial_1 - i\sigma_2\partial_2 \tag{26}$$

where σ_1, σ_2 are the corresponding Pauli matrices. This system is also interesting from a physical viewpoint. It is the effective Hamiltonian of a graphene sheet at low energies.

If we consider periodic boundary conditions, the system can be defined on a 2-D torus $\mathbb{T}^2$. Even in the case of vanishing external magnetic field B it can include non-trivial magnetic fluxes ϵ_1, ϵ_2

$$H_\epsilon = -i\sigma_1(\partial_1 - i\epsilon_1) - i\sigma_2(\partial_2 - i\epsilon_2). \tag{27}$$

This Hamiltonian can also be considered as the Bloch–Floquet decomposition of H defined over the whole plane $\mathbb{R}^2$

$$H = \bigoplus_{\epsilon\in\mathbb{T}^d} H_\epsilon. \tag{28}$$

The spectrum of each Bloch component H_ϵ is in this case

$$E_\epsilon = \pm\sqrt{(n_1-\epsilon_1)^2 + (n_2-\epsilon_1)^2}$$

which gives rise to self-crossing positive and negative energy bands (Fig. 5). In fact this spectrum corresponds to the continuum effective theory of a periodic torus of graphene. The Fermi level $E_F = 0$ corresponds to a filled Dirac sea where all negative energy levels are completely filled.

Since there is no gap between the different spectral bands the system behaves as a conductor as a consequence of its relations to an infinite plate where the charged particles propagate freely.

However, in this case the level crossing is very robust under perturbations. In particular, if we consider the perturbation

$$H_\epsilon^\lambda = H_\epsilon + \lambda(\sin^2 x_1 + \sin^2 x_2). \tag{29}$$

The system (29) is gauge equivalent to the unperturbed Dirac Hamiltonian

$$H_\epsilon^\lambda = e^{i\frac{\lambda}{2}\sigma_1(x_1-\frac{1}{2}\sin 2x_1)} e^{i\frac{\lambda}{2}\sigma_2(x_2-\frac{1}{2}\sin 2x_2)} H_\epsilon^0 e^{i\frac{\lambda}{2}\sigma_1(x_1-\frac{1}{2}\sin 2x_1)} e^{i\frac{\lambda}{2}\sigma_2(x_2-\frac{1}{2}\sin 2x_2)}. \tag{30}$$

Thus in this case the system keeps his conducting character for all periodic perturbations.

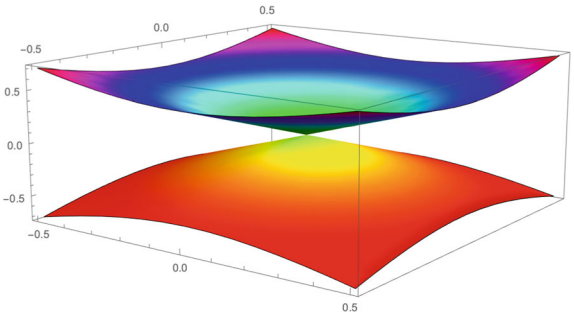

Fig. 5 Low energy levels of a massless Dirac particle on the $\mathbb{T}^2$ torus $-\frac{1}{2} \leq \varepsilon_i \leq \frac{1}{2}, i = 1, 2$. The Fermi level $E_F = 0$ corresponds to a filled Dirac sea. The upper surface represents the conducting band whereas the lower energy bands are completely filled. There are four crossing points in the bands which indicates the quantum transition points $(\varepsilon_1, \varepsilon_2) = (\pm\frac{1}{2}, \pm\frac{1}{2})$, where the energy levels become degenerate (Kramers points)

6 Edge States of Dirac Hamiltonians

Massive Dirac Hamiltonians also appear as the low energy effective Hamiltonians of epitaxially grown graphene on a SiC substrate [19]. If instead of having an infinite sheet which leads via Bloch–Floquet theorem to a system living in a 2D torus, we have a strip with a shorter transversal side of size $2L$, the same theorem tells us that the system can be described by a family of Hamiltonians

$$H_\epsilon = -i\sigma_1(\partial_1 - i\epsilon_1) - i\sigma_2\partial_2 + m\sigma_3, \quad \text{with } \sigma_3 = -i\sigma_1\sigma_2 \tag{31}$$

defined on cylinder of size $2L$ (see Fig. 4) with mass m and boundary conditions at the two ends $x_2 = \pm L$. The boundary conditions which preserve the Hermiticity of the Hamiltonian (31) are essentially of three different types (see [20] for a review of the most general boundary conditions)

(i) Periodic boundary conditions

$$\psi(L) = \psi(-L) \quad \text{and} \quad \partial_2\psi(L) = \partial_2\psi(-L)$$

give rise to a spectrum similar to the torus case (Fig. 6)

$$E_\epsilon = \pm\sqrt{(n_1 - \epsilon_1)^2 + (n_2 - \epsilon_1)^2 + m^2}.$$

In this case the number of modes $\nu_\pm$ with eigenvalues $\pm m$ of the Hamiltonian (31) is $\nu_\pm = 1$ when $\epsilon_1 = \epsilon_2 = 0$ and $\nu_+ = 0$ when either $\epsilon_1 \neq 0$ or $\epsilon_2 \neq 0$ and the result cannot be derived from the the Atiyah–Singer theorem which states that the difference

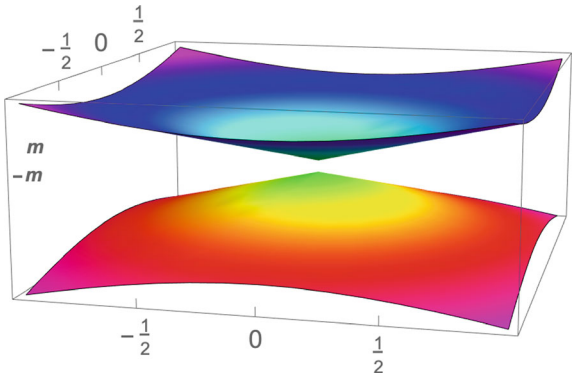

Fig. 6 Low energy levels of the massive Dirac operator with periodic boundary conditions on the $\mathbb{T}^2$ torus for $-\frac{1}{2} \le \varepsilon_i \le \frac{1}{2}, i = 1, 2$. The Fermi level $E_F = 0$ corresponding to a fully filled Dirac sea of negative energy levels. Notice that in this case there is an energy gap $2m = 2$ between the positive energy levels and the Dirac sea levels. However there are again four crossing points of state levels at the quantum transition points $(\varepsilon_1, \varepsilon_2) = (\pm\frac{1}{2}, \pm\frac{1}{2})$, where the band states becomes degenerate (Kramers points)

of the number of zero modes when $m = 0$ with positive ν_+ and negative ν_- chiralities vanishes

$$\nu_+ - \nu_- = 0, \tag{32}$$

because we cannot use the Riemann–Roch formula [21]

$$\dim \mathcal{H}^0 = \frac{1}{8\pi} \int_{T^2} \sqrt{g} R + \dim \mathcal{H}^0_{2g-2}. \tag{33}$$

for getting information about the total number of eigenvalues $\pm m$,

$$\nu_- + \nu_- = \dim \mathcal{H}^0. \tag{34}$$

since the last term in (33) vanishes only for higher genus surfaces $2g - 2 \ge 0$ (Kodaira's vanishing theorem) but not in this case where $g = 1$, as erroneously pointed out in Ref. [22].

(ii) Atiyah–Patodi–Singer (APS) boundary conditions are defined by [23]

$$\bar{\prod}_{\pm} \psi_{S^1_{\pm}} = 0, \tag{35}$$

where $\bar{\prod}_{\pm}$ are the orthogonal projectors to the subspaces $\mathcal{H}^-_{\pm}$ of non-positive eigenvalues of the boundary Dirac operators

$$\not{D}_{S^1_{\pm}} = \mp i\sigma_3(\partial_1 + i\epsilon_1)$$

acting on the Hilbert spaces of boundary spinors $L^2(S^1_\pm)$. The APS boundary conditions are highly non-local. Because of the anti-commutation of σ_2 with the boundary operators $\not{D}_{S^1_\pm}$ we also have that

$$\sigma_2 \Pi^-_\pm \psi_{S^1_\pm} = \Pi^+_\pm \sigma_2 \psi_{S^1_\pm} = 0, \tag{36}$$

i.e. σ_2 acts as an intertwiner operator between the two orthogonal sectors of non-positive and non-negative eigenvalues of $\not{D}_{S^1_\pm}$ in the splitting of the Hilbert space of boundary Dirac spinors $\mathcal{H}_\pm = L^2(S^1_\pm, \mathbb{C}^2) = \mathcal{H}^+_\pm \oplus \mathcal{H}^-_\pm$.

The APS boundary conditions do preserve the chiral symmetry of the massless Dirac operator because the boundary conditions are invariant under chiral symmetry.

$$\sigma_2 \Pi^+_\pm = \Pi^-_\pm \sigma_2.$$

The main reason for choosing APS boundary conditions is that with them the Dirac operator is not only a selfadjoint operator but also elliptic which allow it to satisfy the APS theorem [23].

The APS theorem generalizes the Atiyah–Singer theorem for manifolds with boundaries and establishes that the difference of the numbers of zero modes of the Dirac operator ($m = 0$) with different chiralities can be expressed in analytic terms as

$$\nu_+ - \nu_- = \frac{1}{2}[h(\not{D}_{S^1_+}) - h(\not{D}_{S^1_-})] = 0 \tag{37}$$

where $h(\not{D}_{S^1_\pm}) = \dim \ker \not{D}_{S^1_\pm}$. In particular this implies that there is the same number of modes with energy m than $-m$. However, in the present case it means that there are none.

However in principle there is the possibility of existence of edge states with energies $-m < E < m$ that are localized at the boundaries of the strip. They should be of the form

$$\psi(x_1, x_2) = \mathrm{e}^{in\varphi} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = e^{inx_1} \begin{pmatrix} Ae^{\kappa x_2} + Be^{-\kappa x_2} \\ Ce^{\kappa x_2} + Ce^{-\kappa x_2} \end{pmatrix}. \tag{38}$$

The APS boundary conditions require that

$$\begin{aligned} &\phi_1(\pm L) = 0, \text{for} (n_1 - \epsilon_1) > 0 \\ &(\partial_2) - n_1 + \epsilon_1)\phi_2(\pm L) = 0, \text{for} (n_1 - \epsilon_1) \leq 0. \end{aligned} \tag{39}$$

However, it can be shown that the only eigenvalues of the (31) satisfying these APS boundary conditions must have $\kappa = 0$ which is a delocalized state and not attached to the edges of the strip.

This means that there is no edge state with APS boundary conditions. This is surprising result of the cylinder geometry which is in contrast with the disk topology, where edge states do exist [24].

(iii) Chiral boundary conditions. They are given by

$$\psi(x_1 + 2\pi, x_2) = \psi(x_1, x_2) \tag{40}$$

$$\psi(x_1, \pm L) = \mp i\sigma_2 e^{\theta\sigma_3}\sigma_3\psi(x_1, \pm L). \tag{41}$$

In this case is easy to show that there are edge states with energies $-m \leq E \leq m$ that are localized near the edges of the cylinder.

In order to find these edge states let us consider stationary states of the form

$$\Psi(x_1, x_2) = \mathrm{e}^{inx_1}\begin{pmatrix}\phi_1(x_2)\\ \phi_2(x_2)\end{pmatrix}; \tag{42}$$

satisfying the eigenvalue equation

$$H_\epsilon\Psi(x_1, x_2) = E\Psi(x_1, x_2), \tag{43}$$

and the boundary conditions (41) which now read

$$\phi_2(\pm L) = \pm\mathrm{e}^{\theta}\phi_1(\pm L). \tag{44}$$

Then ϕ_1 and ϕ_2 are solutions of the pair of coupled differential equations,

$$\begin{array}{l}(m - E)\phi_1(x_2) + (n + \epsilon - \partial_2)\phi_2(x_2) = 0\\ (E + m)\phi_2(x_2) - (n + \epsilon + \partial_2)\phi_1(x_2) = 0\end{array} \tag{45}$$

with boundary conditions (44). For positive energies $E > 0$ edge states can be obtained from the ansatz

$$\phi_1(x_2) = A\mathrm{e}^{\kappa x_2} + B\mathrm{e}^{-\kappa x_2} \quad \text{and} \quad \phi_2(x_2) = \frac{n + \epsilon + \partial_2}{E + m}\phi_1(x_2), \tag{46}$$

where the boundary condition (44) determines κ. In the limit of very large values of L the solution is

$$\kappa_\pm = (n \pm \epsilon)\tan\theta + \frac{m}{\cosh\theta} \quad \kappa_\pm > 0. \tag{47}$$

and the energy E of these bound states

$$E_\pm = \frac{n \pm \epsilon}{\cosh\theta} - m\tanh\theta \quad 0 \leq E \leq m. \tag{48}$$

For negative energies $E < 0$

$$\phi_1(x_2) = \frac{n + \epsilon - \partial_2}{-E + m}\phi_2(x_2),$$

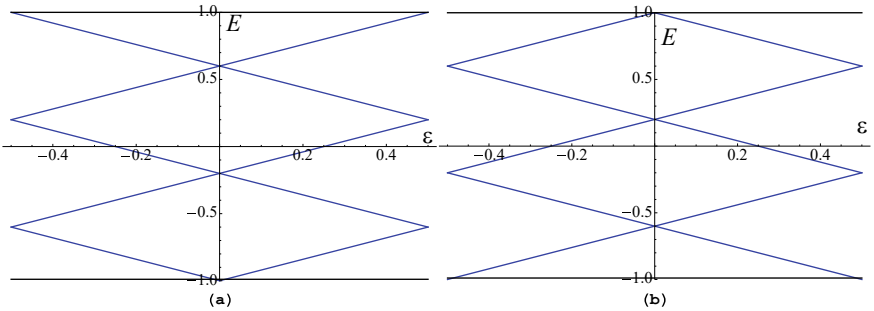

Fig. 7 Energy levels of edge states within the mass gap $-m < E < m$ of the Hamiltonian H_ϵ for two different boundary conditions $\theta = -\log 2$ (case a) and $\theta = \log 2$ (case b). We have assumed unit mass $m = 1$ and $L = \infty$. For any value of the energy within the mass gap there are always two states with the same energy that are localized at opposite edges of the cylinder and move in opposite directions

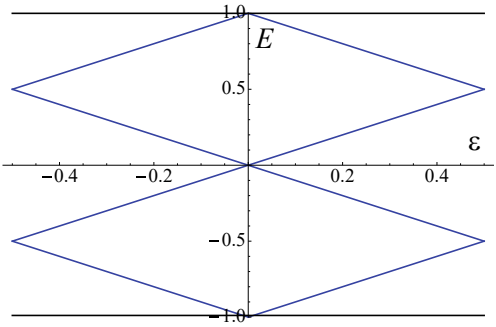

Fig. 8 Energy levels of edge states within the mass gap $-1 < E < 1$ of the Hamiltonian H_ϵ for the symmetric boundary condition $\theta = 0$. We have assumed unit mass $m = 1$ and $L = \infty$. For any value of the energy within the mass gap there are always two states localized at opposite edges of the cylinder moving in opposite directions

κ is given by (47) and the energies of the bound states are

$$E_\pm = \frac{n \pm \epsilon}{\cosh\theta} - m\tanh\theta \quad -m \le E \le 0. \tag{49}$$

The corresponding energy spectrum is displayed in Figs. 7 and 8 as a function of the magnetic flux ϵ for different chiral boundary conditions.

The survival of edge states in the large L limit can be understood by the fact that those states can be localized at each edge of the strip if we displace the location of the strip from $[-L, L]$ to $[0, 2L]$ or $[-2L, 0]$. In both cases $\kappa = m$.

Another interesting property of the system is that for any energy level there are always two edge states that are localized at opposite edges of the cylinder and move in opposite directions. The extra degeneracy is due to the symmetry of the Hamiltonian

under the anti-unitary generalization of parity defined by

$$\mathcal{P}\Psi(x_1, x_2) = \sigma_3 \Psi(x_1, -x_2)^*. \tag{50}$$

In the case of parity symmetric boundary conditions $\theta = 0$ and vanishing magnetic flux $\epsilon = 0$, for large values of L the positive energy level $E = n > 0$ given by

$$\Psi(x_1, x_2) = \mathrm{e}^{inx_1 + mx_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \tag{51}$$

is localized on the right edge of the cylinder, whereas the state with the same energy $E = n > 0$ given by

$$\Psi(x_1, x_2) = \mathrm{e}^{inx_1 - mx_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{52}$$

is localized at the opposite edge.

This is general property for all states within the gap $-m < E < m$ which is responsible for the existence of edge currents. The presence of a magnetic flux across the cylinder does not change this property.

On the other hand the phenomenon is highly dependent on the type of boundary condition considered. When $\theta \to \infty$ the ϕ_1 component satisfies Dirichlet boundary conditions, whereas in the limit $\theta \to -\infty$ it is ϕ_2 which vanishes at the edges. These boundary conditions are very similar to the zig-zag boundary conditions of graphene. Another popular boundary condition is $\theta = 0$. In this case $\phi_2 = \phi_1$ which in graphene corresponds to the armchair boundary conditions. However, in the present case with a non-trivial mass term the edge states disappear in both limits, breaking down the conducting properties of the model. For finite but large values of θ, there are always edge states but they are so much tied to the edges that the become delta-like states that disappear from the Hilbert space and the spectrum [17].

The number of edge states depends on how large is the mass gap $2m$. For larger masses, there is a higher number of edge states.

However, not all crossings of edge states in the large $L \gg \dfrac{1}{m}$ limit are very robust. For lower values of the transverse length of the strip there is always a gap inside the edge spectrum as it is shown in Figs. 9 and 10. This level repulsion is due to the fact that in spite of the fact that the system is time-reversal invariant at the Kramers points, the realization of the symmetry is not anti-unitary and therefore it does not imply the existence degeneracy of energy levels at those points. The situation will change if we include spin degrees of freedom which can give rise to topological insulators [3, 4].

There is a connection between the Dirac operator (31) and the Schrödinger Hamiltonian

$$H^m_\epsilon = -\frac{1}{2}\left(\partial_1 - i\epsilon_1\right)^2 - \frac{1}{2}\left(\partial_2 - i\epsilon_2\right)^2 + m^2. \tag{53}$$

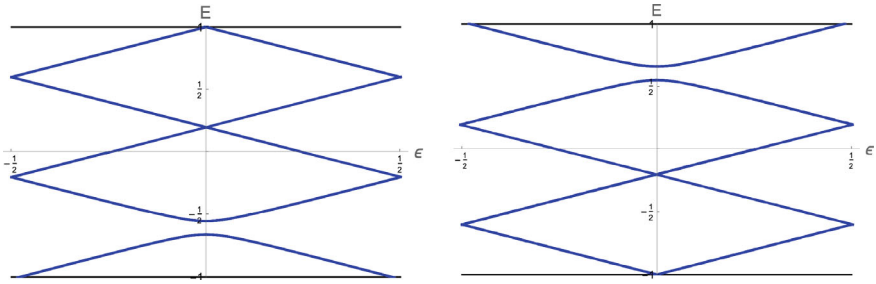

Fig. 9 Energy levels of edge states within the mass gap $-m < E < m$ of the Hamiltonian H_ϵ for two different boundary conditions $\theta = -\log 2$ (case a) and $\theta = \log 2$ (case b). We have assumed unit mass $m = 1$ and $L = 2$. Notice that in these case there is a gap in the spectrum of edge states at the Kramers point $\epsilon = 0$. This gap vanishes in the limit $L \to \infty$, but for any finite size opens the possibility of having a bulk-edge insulator. For any value of the energy outside the gap there are always two states with the same energy that are localized at opposite edges of the cylinder and move in opposite directions. The crossings at Kramers points $\epsilon = \pm\frac{\pi}{2}$ persist for any values of m and L

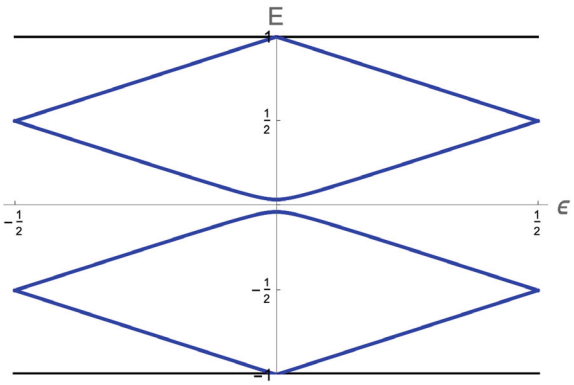

Fig. 10 Energy levels of edge states within the mass gap $-m < E < m$ of the Hamiltonian H_ϵ for the symmetric boundary condition $\theta = 0$. We have assumed unit mass $m = 1$ and $L = 2$. Notice that in these case there is a gap in the spectrum of edge states. For any value of the energy outside this gap there are always two states localized at opposite edges of the cylinder moving in opposite directions. For Fermi energy values inside the gap the system is absolute insulator both in the bulk and the edge

Indeed,

$$H_\epsilon^2 = H_\epsilon^m \tag{54}$$

The question is which boundary conditions of H_ϵ^m do correspond to the chiral boundary conditions (41) of H_ϵ?

The essential extra requirement is that $H_\epsilon \Psi$ has also to satisfy the chiral boundary condition. After some algebra one get that Ψ has to satisfy the extra boundary

condition

$$\partial_2 \Psi(\pm L) = -i\sigma_3 \ \tanh(\sigma_3\theta)\sigma_3\partial_1\Psi(\pm L) \tag{55}$$

to be in domain of the Schrödinger Hamiltonian H_ϵ^m. Notice that with these boundary conditions the edge states of H_ϵ^m are not negative in contrast with what happens with most of its boundary conditions for small values of m [24]. In summary, the boundary conditions induced by chiral boundary conditions belong to a one parameter family which preserves the lower bound m of the spectum of the Schrödinger Hamiltonian H_ϵ^m.

In the case of periodic boundary condition this behaviour is natural because both Hamiltonians cannot have edge states by construction. In this particular case the connection between Dirac and Schrödinger Hamiltonians can be extended for periodic perturbations.

7 Integer Hall Effect

The Hall effect plays a very important historical role in the path leading to the discovery of topological insulators. The effect is driven by the dynamics of a quantum charge e moving in a two dimensional plate under the effects of a constant transverse magnetic field. The corresponding Hamiltonian is

$$H_A \equiv -\frac{1}{2m}\left[(\partial_1 - ieA_1)^2 + (\partial_2 - ieA_2)^2\right] \tag{56}$$

where m is the mass of the charged particle and $\mathbf{A} = (A_1, A_2)$ is the vector potential associated to the magnetic field,

$$\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, B). \tag{57}$$

which we assume is pointing along the OZ axis direction.

With particular choice of the vector potential

$$\mathbf{A} = (-\tfrac{B}{2}x_2, \tfrac{B}{2}x_1, 0) \tag{58}$$

the Hamiltonian reads

$$H_A \equiv -\frac{1}{2m}\left[(\partial_1 + ie\tfrac{B}{2}x_2)^2 + (\partial_2 - ie\tfrac{B}{2}x_1)^2\right]. \tag{59}$$

Again using the translation invariance of the magnetic field and Bloch–Floquet theorem we can reduce the problem to a family of Hamiltonians of the form

$$H_A^\epsilon \equiv -\frac{1}{2m}\left[(\partial_1 + ie\tfrac{B}{2}x_2 - i\epsilon_1)^2 + (\partial_2 - ie\tfrac{B}{2}x_1 - ie\epsilon_2)^2\right] \tag{60}$$

restricted to a two dimensional torus $\mathbb{T}^2 = (0, 2\pi) \times (0, 2\pi)$ where the $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$ parameters are defined on the dual torus $\widehat{\mathbb{T}}^2 = (-1/2, 1/2) \times (-1/2, 1/2)$ which is the Brillouin zone of the plate.

Quantization consistency requires the quantization of the magnetic flux

$$\frac{1}{2\pi} \int_{\mathbb{T}^2} eB \, d^2\mathbf{x} = k \in \mathbb{Z}.$$

In such a case the quantum Hamiltonian (60) is a Hermitian operator acting on quantum states that are sections of a line bundle $E(\mathbb{T}^2, \mathbb{C})$ with first Chern number $c_1(E) = k$. If $k \neq 0$ this bundle is non-trivial and there is no non-vanishing global section. One can choose a dense subset of $(0, 2\pi) \times (0, 2\pi) \in \mathbb{T}^2$ where the quantum states satisfy the boundary conditions

$$\begin{aligned} \psi(x_1 + 2\pi, x_2) &= \mathrm{e}^{i\frac{k}{2}x_2} \psi(x_1 + 2\pi, x_2) \\ \psi(x_1, x_2 + 2\pi) &= \mathrm{e}^{-i\frac{k}{2}x_1} \psi(x_1, x_2 + 2\pi). \end{aligned} \tag{61}$$

The eigenvalues of H_A^{ϵ} are the Landau levels

$$E_n = \frac{|k|}{2\pi m} \left(n + \frac{1}{2} \right) \qquad n = 0, 1, 2, \ldots, \tag{62}$$

all of them with the same degeneracy $|k|$. Notice that the spectrum is independent of the ϵ parameters of the magnetic vector potential (58).

The states of the first Landau level can be given in terms of holomorphic sections (theta functions [21, 25]) of the line bundle $E(\mathbb{T}^2, \mathbb{C})$ with respect to the complex structure induced by the vector potential A, and the complex structure of the torus $\mathbb{T}^2$,

$$\psi_0^l(\boldsymbol{\epsilon}, \mathbf{x}) = \left(8\pi^4 k\right)^{-\frac{1}{4}} \mathrm{e}^{i\frac{k}{4\pi}(x_1 + 2\epsilon_1)x_2} \sum_{m=-\infty}^{\infty} \mathrm{e}^{im\left(x_1 + \epsilon_1 + 2\pi\frac{l}{k}\right) - \frac{1}{4\pi k}(2\pi m + k x_2 + k\epsilon_2)^2}, \tag{63}$$

where $l = 0, 1 \ldots, |k| - 1$, $\mathbf{x} = (x_1, x_2)$ and $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$. Notice that all them have nodal points as it must be since they are sections of a non-trivial vector bundle [26].

This structure of the Landau spectrum is responsible for the integer quantum Hall effect. In the large $k \gg 1$ limit the electrons can only fill some low energy Landau levels giving rise to different regimes of electric conductivity. As we will show later the quantization of the Hall conductivity σ_{xy} is one of the essential features of the integer quantum Hall effect.

8 Integer Hall Effect on Graphene

If we apply a constant transverse magnetic field to a graphene sheet the dynamics is effectively driven by the Hamiltonian

$$H_D = -i\sigma_1(\partial_1 + ieB/2) - i\sigma_2(\partial_2 - ieB/2) \tag{64}$$

In this case the Bloch–Floquet theorem tell us that the spectral problem can be reduced to that of a family of Dirac Hamiltonians

$$H_D^\epsilon = -i\sigma_1(\partial_1 - i\epsilon_1 + ieB/2) - i\sigma_2(\partial_2 - i\epsilon_2 - ieB/2)) \tag{65}$$

defined on the torus $\mathbb{T}^2$, parametrized by the Bloch parameters $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) \in \widehat{\mathbb{T}}^2$ of the dual torus $\widehat{\mathbb{T}}^2 = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ of the Brillouin zone of the crystal.

The spectrum of this Hamiltonian can be found using the results of the previous section. Indeed, the square of the Dirac Hamiltonian is related to the standard non-relativistic Hamiltonian of the integer Hall effect (65) by the identity

$$(H_D^\epsilon)^2 = 2mH_A^\epsilon + eB\sigma_3. \tag{66}$$

Thus, the spectrum of (66) is

$$E_n^\pm = 2eB(n + \tfrac{1}{2}) \pm eB = 2eBr_\pm, \tag{67}$$

where $r_+ = n + 1$ and $r_- = n$. Thus, the energy spectrum of H_D^ϵ is $E_n^\pm = \pm\sqrt{2eBr}$, $r \in \mathbb{N}$ and the degeneracy is $2k$ for $r > 0$ and k for $r = 0$.

The quantization of the spectrum is the basis of the integer quantum Hall effect. The existence of discretized energy levels implies the insulating character of the metal due to the effect of the magnetic field. However, in finite size metals the situation can change due to the action of the edges. What is remarkable is that even in absence of edges it is possible to infer the quantization of the Hall conductivity [6].

It has been shown that the zero modes can be identified with holomorphic sections of a holomorphic bundle with Chern class k over the torus $\mathbb{T}^2$. From this identification we can infer that the dimension of the space of quantum states in the first Landau level is given by the Riemann–Roch formula

$$\dim \mathcal{H}_k^0 = \frac{1}{8\pi}\int_{\mathbb{T}^2} \sqrt{g}R + \frac{1}{2\pi}\int_{\mathbb{T}^2} eB + \dim \mathcal{H}_{-k}^0 = k \tag{68}$$

because the torus is flat $R = 0$ and since $(k \geq 1)$ the last term in (33) vanishes by Kodaira's vanishing theorem [21].

The explicit calculation of these zero modes which define the first Landau level of the crystal gives

$$\boldsymbol{\psi}_0^{l_+} = \begin{pmatrix} \psi_0^{l_+}(\boldsymbol{\epsilon}, \mathbf{x}) \\ 0 \end{pmatrix} \quad \text{for } k > 0, \qquad \boldsymbol{\psi}_0^{l_-} = \begin{pmatrix} 0 \\ \psi_0^{l_-}(\boldsymbol{\epsilon}, \mathbf{x}) \end{pmatrix} \quad \text{for } k < 0, \tag{69}$$

where $\psi_0^{l_\pm}$ is given by (63).

The Fourier-Mukai transform [27–30] associates to the abelian gauge field $\mathbf{A}$ of the magnetic field B a non-abelian $SU(k)$ gauge field

$$A_j^{l,l'} = i \int_{\mathbb{T}^2} d^2\mathbf{x}\, \boldsymbol{\psi}_0^{l\dagger} \partial_{\epsilon_j} \boldsymbol{\psi}_0^{l'}, \tag{70}$$

on the dual torus $\widehat{\mathbb{T}}^2$ via the family zero modes. This gauge field is a non-abelian version of the Berry phase [31] and the corresponding Chern class equals the rank of the original gauge field, in this case $rank\, U(1)$=1, i.e.

$$c_1(A_i^{l,l'}) = i \frac{1}{2\pi} \int_{\widehat{\mathbb{T}}^2} d^2\boldsymbol{\epsilon} \int_{\mathbb{T}^2} d^2\mathbf{x} \left[(\partial_{\epsilon_1} \boldsymbol{\psi}_0^l)^\dagger \partial_{\epsilon_2} \boldsymbol{\psi}_0^l - (\partial_{\epsilon_2} \boldsymbol{\psi}_0^l)^\dagger \partial_{\epsilon_1} \boldsymbol{\psi}_0^l \right] = 1.$$

This rank-Chern number duality is one of the characteristics of the Fourier-Mukai transform. In fact this property can be extended to higher Landau levels and this is what Thouless et al. [6, 7] used to prove that the conductivity given by the Kubo formula

$$\sigma_{xy}^k = -\frac{ie^2}{4\pi^2} \sum_{n=0}^{\nu} \sum_{l=0}^{|k|-1} \int_{\widehat{\mathbb{T}}^2} \int_{\mathbb{T}^2} \left[(\partial_{\epsilon_1} \boldsymbol{\psi}_n^l)^\dagger \partial_{\epsilon_2} \boldsymbol{\psi}_n^l - (\partial_{\epsilon_2} \boldsymbol{\psi}_n^l)^\dagger \partial_{\epsilon_1} \boldsymbol{\psi}_n^l \right] = \frac{e^2}{2\pi} \nu \tag{71}$$

is quantized. Here ν is the highest integer with $E_\nu < E_F$, i.e. the number of Landau levels which are completely filled.

9 Hall Conductivity and Edge States

In the previous section we have seen that to understand the Hall conductivity is enough to analyze the contribution of the bulk states $\boldsymbol{\psi}_n^l$ (71). However, from a physical viewpoint all realistic materials are finite-dimensional and their behaviour depends on their boundary conditions.

For a very long plate strip one can assume that it is infinite along the longitudinal direction which leads via Bloch–Floquet theorem to a a family of systems living in a 2D cylinder. In the non-relativistic case the corresponding Hamiltonian (59) of the plate can be reduced to a family of Hamiltonians (60) defined over the cylinder. But the Hermiticity condition of both Hamiltonians requires the introduction of suitable boundary conditions (20). In this case there is no need of quantization of the magnetic flux.

The spectrum of the Hamiltonian (60) can be obtained in a similar way as in the torus. The only difference is that in stead of plane waves we have to use hypergeometric functions to match the boundary conditions. Indeed, if we perform an unitary transformation

$$2m e^{-ieBx_1x_2/2} H_A^\epsilon e^{ieBx_2x_1/2} = \frac{1}{2m}\Delta_A^\epsilon, \tag{72}$$

where

$$\Delta_A^\epsilon \equiv -(\partial_1 + ieBx_2 - i\epsilon_1)^2 - \partial_2^2. \tag{73}$$

the most general solution of the spectral equation $\Delta_\epsilon \varphi = 2mE\varphi$ can be expressed as a linear combination of confluent hypergeometric functions [32]

$$\psi_E(x_1, x_2) = e^{ikx_1}\left(A_1\xi_1(x_2) - A_2\xi_2(x_2)\right). \tag{74}$$

with

$$\xi_1(n, x_2) = e^{-\frac{1}{4}x^2}\, {}_1F_1\Big(-\frac{mE}{eB} + \frac{1}{4}, \frac{1}{2}; \frac{x^2}{2}\Big) \tag{75}$$

$$\xi_2(n, x_2) \;= x\, e^{-\frac{1}{4}x^2}\, {}_1F_1\Big(-\frac{mE}{eB} + \frac{3}{4}, \frac{3}{2}; \frac{x^2}{2}\Big) \tag{76}$$

where ${}_1F_1(a, c; x)$ is the *confluent hypergeometric* function [33] and

$$x = \sqrt{2eB}\Big(\frac{n - \epsilon_1}{eB} \;+\; x_2\Big).$$

Now the energy spectrum is derived from the boundary conditions constraints. Let us consider only three types of boundary conditions.

(i) Dirichlet boundary conditions $\alpha = \pi, \beta = 0, \quad \psi(\pm L) = 0$
(ii) Newman boundary conditions $\alpha = 0, \beta = 0, \quad \partial_{x_2}\psi(\pm L) = 0$
(iii) Robin boundary conditions $\beta = 0, \quad \partial_{x_2}\psi(\pm L) = \pm \tan\frac{\alpha}{2}\psi(\pm L)$.

The matching of boundary conditions at $x_2 = -L$ can be guaranteed by the choices

$$\begin{aligned}
e^{-inx_1}\psi_E^{\mathrm{D}}(x_1; x_2) &= \xi_2(n; -L)\xi_1(n, x_2) - \xi_1(n; -L)\xi_E^2(n, x_2) \quad \text{(Dirichlet)}\\
e^{-inx_1}\psi_E^{\mathrm{N}}(x_1; x_2) &= \xi_2{}'(n; -L)\xi_1(n, x_2) - \xi_1{}'(n; -L)\xi_2(n; x_2) \quad \text{(Neumann)}\\
e^{-inx_1}\psi_E^{\mathrm{R}}(x_1; x_2) &= \chi_2^-(n, \alpha)\xi_1^\alpha(n, x_2) - \chi_1^-(n, \alpha)\xi_2(n, x_2) \qquad \text{(Robin)}
\end{aligned}$$

where

$$\chi_i^\mp(n, \alpha) \equiv \xi_i{}'(n; \mp L) \pm \tan\frac{\alpha}{2}\,\xi_i(n; \mp L).$$

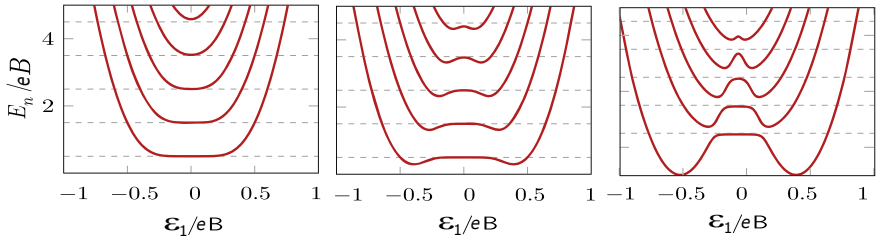

Fig. 11 Dispersion diagrams of the integer quantum Hall effect in the rescaled variables E/eB and ϵ_1/eB for different boundary conditions [32]; from left to right: Dirichlet, Neumann and $\tan\alpha L = 10$ Robin boundary conditions (for $eB = 50$)

Matching the boundary conditions at $x_2 = L$ requires the vanishing of the following spectral functions

$$
\begin{aligned}
h_n^{\mathrm{D}}(E) &= \xi_2(n;-L)\xi_1(k;L) - \xi_1(n;-L)\xi_2(k;L) \quad \text{(Dirichlet)}\\
h_n^{\mathrm{N}}(E) &= \xi_2{}'(n;-L)\xi_1{}'(k;L) - \xi_1{}'(n;-L)\xi_2{}'(n;L) \quad \text{(Neumann)}\\
h_n^{\mathrm{R}}(E) &= \chi_2^-(n,\alpha)\chi_1^+(n,\alpha) - \chi_1^-(n,\alpha)\chi_2^+(n,\alpha) \quad \text{(Robin)}
\end{aligned}
$$

i.e. the zeros of these spectral functions give us the energy spectrum of the Haniltonian. These solutions can be found numerically and they are displayed in Fig. 11 for different types of boundary conditions [32].

In summary the edge states change the insulating behaviour of the graphene strip to a conducting one where the currents are located at the edges of the strip. We also observe a peculiar behaviour of such a currents for Neumann and Robin boundary conditions where the energy spectra of edge states is not a convex function. There are bumps around the Kramers points $\epsilon_1/eB = 1/2$. Those bumps imply a change of direction of the edge currents.

10 Hall Effect in Graphene Strips

The Hall effect on a finite size graphene strip is highly dependent on the boundary conditions of the graphene which essentially depend on how the boundary links of the graphene honeycomb lattice are cutted.

10.1 APS Boundary Conditions

If we impose APS boundary conditions on the Hamiltonians (65) we obtain a spectrum which differs from the one given for periodic boundary conditions on the torus as in the non-relativistic case (60).

However, there are some differences which are worth to point out. The first one can be derived from the APS theorem that generalizes the Atiyah–Singer theorem for manifolds with boundaries. It states in this case that the difference of the number of zero modes of (65) with positive ν_+ and negative ν_- chiralities is given by the analytic formula

$$\nu_+ - \nu_- = \frac{1}{2\pi}\int_{[-L,L]\times S^1} B + \frac{1}{2}[h(\not{D}_{S^1_+}) - h(\not{D}_{S^1_-} + \eta(\not{D}_{S^1_+}) - \eta(\not{D}_{S^1_-})], \quad (77)$$

where $h(\not{D}_{S^1_\pm}) = \dim \ker \not{D}_{S^1_\pm}$, and $\eta(\not{D}_{S^1_\pm})$ is the eta invariant of the Dirac operator along the boundaries $\not{D}_{S^1_\pm}$ given by its spectral asymmetry, which formally is defined by

$$\eta(\not{D}_{S^1_\pm}) = \sum_{\lambda\in \mathrm{Sp}\,\not{D}_{S^1_\pm}} \mathrm{sign}\,\lambda = \sum_{\lambda\in \mathrm{Sp}\,\not{D}_{S^1_\pm}} \frac{\lambda}{|\lambda|}. \quad (78)$$

The kernel of $\not{D}_{S^1_-}$ is trivial ($h(\not{D}_{S^1_-}) = 0$) whenever $\epsilon_2 \notin \mathbb{Z}$ and one-dimensional ($h(\not{D}_{S^1_-}) = 1$) otherwise. In a similar manner $h(\not{D}_{S^1_+})$ also vanishes whenever $\epsilon_2 + 2\pi eB \notin \mathbb{Z}$ and $h(\not{D}_{S^1_+}) = 1$ otherwise. The calculation of the η invariants is more involved. There are many ways of obtaining the value η, by using different regularization procedures [34–36].

If we ignore the existence of ambiguities in some very peculiar regularization methods [34–36], most of the methods give the same results [16]

$$\eta(\not{D}_{S^1_-})] = \epsilon_2 - [\epsilon_2], \qquad \eta(\not{D}_{S^1_+})] = \epsilon_2 + 2\pi eB - [\epsilon_2 + 2\pi eB], \quad (79)$$

when $\epsilon_2 \notin \mathbb{Z}$ and $\epsilon_2 + 2\pi eB \notin \mathbb{Z}$. The brackets $[\epsilon]$ meaning the integer part of ϵ.

Now by Stokes theorem

$$\int_{[-L,L]\times S^1} F = e\int_{S^1_+} A - e\int_{S^1_+} A = 2\pi eB, \quad (80)$$

which implies that

$$\nu_+ - \nu_- = [\epsilon_2 + 2\pi eB] - [\epsilon_2]. \quad (81)$$

Since $\nu_\pm = 0$ when $\mp B \geq 0$, the total number of zero modes of the Dirac operator is

$$\nu = \nu_+ + \nu_- = [\epsilon_2 + 2\pi eB] - [\epsilon_2] = |[\epsilon_2 + 2\pi eB] - [\epsilon_2]|. \quad (82)$$

Notice that the number of zero modes is always a positive integer even when the magnetic flux crossing the plate is not an integer multiple of the Bohr quantum as in the periodic boundary conditions case. To some extent the contribution of the boundary supplements the external magnetic flux to reach the nearest integer quantum value. This result is also independent of the ambiguities that arise in the calculation of the spectral asymmetry η of the boundary Dirac operators [36].

In summary the spectrum of the Dirac operator with APS boundary conditions keeps a highly degenerated ground state (first Landau level) as for periodic boundary conditions, but in addition it has some edge states whose lowest energy equals that of the second Landau level. [11, 16]. Thus, the lowest energy edge states opens gap which keeps the plate as a genuine bulk-edge insulator.

Notice that since chiral symmetry is unbroken by the APS boundary conditions there is a paring between positive and negative eigenvalues of the spectrum of the Dirac operator $\not{D}_A$. In the lowest Landau level all eigenstates have the same chirality as the sign of the magnetic field B, whereas the nearest edge states have the opposite chirality. We also remark the bending of the high energy levels with respect to their values for an infinite plate. However, it is quite remarkable as already pointed out that the APS theorem guarantees that there is no bending for the ground state.

10.2 Chiral Boundary Conditions

From a physical viewpoint APS boundary conditions are not realistic for ordinary metals. Unlike the APS boundary conditions physical boundary conditions are local, that is they are defined by local constraints induced by local properties of the corresponding boundaries. Among the local boundary conditions the simplest ones are the chiral boundary conditions that are given by (41), i.e.

$$\begin{pmatrix} \mathbb{I} + i\sigma_2 e^{\theta\sigma_3}\sigma_3 & 0 \\ 0 & \mathbb{I} - i\sigma_2 e^{\theta\sigma_3}\sigma_3 \end{pmatrix} \begin{pmatrix} \psi_{s^1_+} \\ \psi_{s^1_-} \end{pmatrix} = 0. \tag{83}$$

Taking into account the relation given by the algebraic identity (66) between the square of the Dirac operator and the non-relativistic Hamiltonian H^ϵ_A one can find the spectrum of H^ϵ_D from that obtained for H^ϵ_A in the previous section,

The only problem is to select the right boundary conditions of Δ^ϵ_A which are induced by the chiral boundary condition of H^ϵ_D. A simple calculation shows that they are given by

$$\partial_2 \mathbf{\Psi}(x_1, -L) = \pm i\sigma_3 \, \tanh(\sigma_3\theta) \, \partial_1 \mathbf{\Psi}(x_1, L) \tag{84}$$

In the case of $\theta = -\infty$ the left component of $\mathbf{\Psi}$ satisfies the Dirichlet boundary condition $\psi_1 = 0$ whereas the lower component satisfies the Neumann boundary condition $\partial\psi_2 = 0$. Therefore, in this case the spectrum is given by the left graphic of Fig. 11. In any case the other boundary conditions do not give rise to bumps like the ones observed in the non-relativistic case of Fig. 11 because the corresponding boundary conditions do not correspond to the ones induced from chiral boundary conditions of the Dirac operators H^ϵ_D.

11 Bulk-Edge Correspondence

The action of a constant electric field E along the axis of the cylinder modifies the Hamiltonian[1]

$$H_A^E = \tfrac{1}{2m}\Delta_A^\epsilon + eEx_2$$

introducing an slight tilt into its spectrum (see Fig. 12) which mixes the spectral bands and generates a current along the cylinder.

The Hall conductivity

$$\sigma_{xy} = \frac{I_\varphi}{2\pi E}$$

measures the ratio of the induced transverse current I_φ per unit area and the applied electric field E. The transverse current is given by the sum over all occupied states with energies below the Fermi level E_F, i.e. over all eigenstates ψ_{nj} of H_A^E with energies $E_{nj} < E_F$

$$I_\varphi = -\frac{ie}{m}\sum_{n,j}^{E_{nj}<E_F}\int d\epsilon_1 \int_{[-L,L]\times S^1} \psi_{nj}^*(\partial_\varphi + i\epsilon_1)\psi_{nj} = e\sum_{n,j}^{E_{nj}<E_F}\int d\epsilon_1 \int_{[-L,L]\times S^1} \psi_{nj}^*(\partial_{\epsilon_1} H_A^E)\psi_{nj},$$

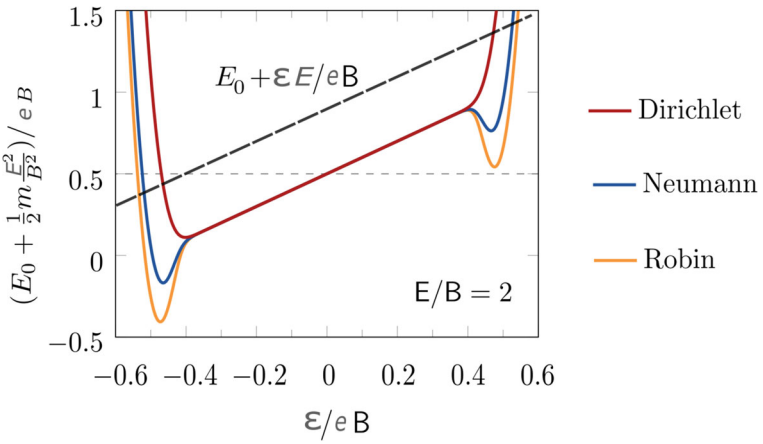

Fig. 12 Effect of different boundary conditions on the first energy band $E_0(\epsilon)$ of the integer quantum Hall effect when an electric field E is applied across the shorter part of the strip at the same time that of a magnetic field B perpendicular to the plate; Here we assumed the values $B = 500$ and $E = 1000$

[1] From here on we assume a non-relativistic approximation to the interaction with the electric field which is closer to the phenomenological setup of the quantum Hall effect.

where we have used the same trick leading to identity (30). Now, since the eigenstates ψ_{nj} are normalized,

$$\int_{[-L,L]\times S^1} (\partial_{\epsilon_1}\psi^*_{nj}) H^E_A \psi_{nj} + \int_{[-L,L]\times S^1} \psi_{nj} H^E_A \partial_{\epsilon_1}\psi_{nj} = 0. \tag{85}$$

Thus,

$$I_\varphi = e \sum_{n,j}^{E_{nj}<E_F} \int d\epsilon_1 \partial_{\epsilon_1} \int_{[-L,L]\times S^1} \psi^*_{nj} H^E_A \psi_{nj} = e \sum_{n,j}^{E_{nj}<E_F} \int d\epsilon_1 \partial_{\epsilon_1} E_{nj}, \tag{86}$$

where $E_n : S^1 \to \mathbb{R}$, $n = 0, \ldots, \infty$ are the real functions which describe the spectrum of Landau levels of H^E_A on the cylinder for different values of the transverse magnetic flux $\epsilon_1 \in S^1$, and the intensity of the Hall current is

$$I_\varphi = e \sum_{n=0}^{\nu_F} \int_{\epsilon_n^-}^{\epsilon_n^+} d\epsilon_1 \partial_{\epsilon_1} E_n, \tag{87}$$

where $\epsilon_n^\pm$ are the two values of the magnetic flux where $E_n(\epsilon_n^\pm) = E_F + eE(\epsilon_+ - \epsilon_-)$ and ν_F is the number of filled Landau levels below the Fermi level. Here is where the differences between the different types of boundary conditions appear. In the case of chiral boundary conditions (41) the curves $E_n(\epsilon_2)$ are smooth due to the fact that the transition from edge to bulk states is continuous [37], and the Hall current is

$$I_\varphi = e \sum_{n=0}^{\nu_F} \int_{\epsilon_n^-}^{\epsilon_n^+} d\epsilon_1 \partial_{\epsilon_1} E_n = e \sum_{n=0}^{\nu_F} [E_n(\epsilon_n^+) - E_n(\epsilon_n^-)] = e^2 E \sum_{n=0}^{\nu_F} (\epsilon_n^+ - \epsilon_n^-). \tag{88}$$

Thus, we obtain a similar formula for quantized conductivity that in the bulk approach (71). However, in this case the contribution of both bulk and edge states is essential to keep the Hall conductivity quantization constant for any value of the applied magnetic field B.

The main effect of edge states is to correct the integer quantization of ν_F, specially at the transitions between two Hall plateaux [32].

The non-relativistic Hamiltonian H^E_A can be related with the square of Dirac Hamiltonian of a graphene ribbon. If we consider the boundary conditions of Δ_A induced from APS boundary conditions of the corresponding effective Dirac Hamiltonian operator of the graphene ribbon the derivation of the quantization of the conductivity is not so straightforward. Indeed APS boundary conditions give rise to discontinuities in the spectral curves indicating the the sharp distinction between edge a bulk states [11]. In particular, the degeneracy of bulk states in the first Landau level is maximal thanks to the APS index theorem, whereas the first boundary states appear after a spectral gap. In this case the formula (88) does not holds, due to the

extra contributions of the discontinuities. In consequence, the quantization of the conductivity under APS boundary conditions does not follow the same pattern as that of chiral boundary conditions.

In some sense this result is the counterpart of the APS index theorem, which holds for APS boundary conditions, whereas it does not make any sense for chiral boundary conditions.

In summary, the bulk-edge correspondence in the integer Hall effect is highly dependent on the boundary conditions and is never exact, because in general $(\epsilon_n^+ - \epsilon_n^-) \neq \nu_F \in \mathbb{N}$ which is in agreement with the experimental results.

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Index

A
$\mathcal{A}\mathbb{P}_1$, 198

B
Banach algebra, 11
Biorthogonal system, 219
Borel subgroup (of $\mathcal{U}_{\rho'}$), 151
Bott periodicity, 22
Boundary conditions
  Atiyah–Patodi–Singer (APS), 260
  chiral, 273
  periodic, 223, 258
  pseudoperiodic, 253
Brillouin zone, 252
Bulk-edge correspondence, 274
Bundle
  coefficient-, 171
  cotangent, 9
  frame, 81, 84
  normal, 9
  principal, 62, 63, 65
  tangent, 8, 59, 173
  tautological, 141, 170
  vector, 6

C
Chern class, 268
Coefficient bundle, 171
Cohomology, 114, 115
Complex structure, 88, 91, 176
  almost, 88
Connection, 61, 96
  flat, 65, 67, 75
  Levi–Civita, 97
  symmetric, 97
Convolution, 30
Covariant derivative, 127, 156
C^*-algebra, 11
Curvature, 102

D
Degree, 65–67, 70, 72, 77, 78
Developing map, 63
Dirac operator, 258
Distribution
  on a manifold, 36
$\mathcal{D}$ (Poincaré disk of a unital C*-algebra), 121

E
Edge state, 256
Euler characteristic, 41, 60, 75, 76, 78, 79, 102
Euler class, 65
Exact couple, 105

F
Finsler
  distance, 129
  metric, 128, 159
Foliation, 45
  Reeb, 47
Fourier
  coefficient, 221
  series, 221
Fredholm index, 22

G
$G_{\mathcal{A}}$, 125
Gelfand transform, 14

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Geodesic, 100, 127
G^+, 125

H
Hall effect, 266
Holonomy, 50
Homogeneous space, 137

I
Immersion, 4
Index
  analytic, 35
  Fredholm, 22
  topological, 53
Involution, 11
Isotropy group, 137

K
$\mathcal{K}_\rho$, 137
K-theory, 19

L
$\mathfrak{L}$-admissible operator, 239
Laplace–Beltrami operator, 250
$\mathfrak{L}$-convolution, 236
$\mathfrak{L}$-Fourier transform, 234
Lie
  algebroid, 27
  groupoid, 25
Liouville, 195
Longitudinal index theorem for foliations, 55

M
Manifold, 3
  affine, 5, 60
    complete, 64, 80
    special, 84
  Hopf, 61
  smooth, 4
Moment map, 181
Morita equivalence, 23

P
Para-complex structure, 88
Para-hypercomplex structure, 88, 89
Pfaffian, 103
Poincaré disk (of a unital C*-algebra), 121
Poincaré half-space (of a unital C*-algebra), 122
Polar decomposition, 130
Pontryagin dual, 33
Presheaf, 111
Pseudo-differential operator, 36, 226
  global, 229

Q
$\mathcal{Q}_\rho$, 121

R
Reductive homogeneous space, 137, 142
Reeb foliation, 47
Riemannian form, 180
Riesz
  basis, 219
  system, 223

S
Sheaf, 111, 112
Sobolev spaces, 224
Spectral sequence, 104
Spectrum, 12
Submanifold, 5
Submersion, 4
Symbol, 226
  classical, 37
  global, 228
  principal, 38
Symmetry (of a C*-algebra), 131
Symplectic form, 179

T
Tangent bundle, 8
Tensor
  curvature, 102
  Nijenhuis, 87
  torsion, 97
Theorem
  Atiyah-Singer, 53
    for families, 54
  Bott periodicity, 22
  Chern–Gauss–Bonnet, 104
  Frobenius, 48
  Gelfand–Mazur, 13
  Hopf degree, 77
  Klingler, 85, 95
  Kostant–Sullivan, 80, 84
  Levi–Civita, 98

Longitudinal index for foliations, 55
Peter-Weyl, 227
Serre–Swan, 19
Smillie, 76, 78
Stone-Weierstrass, 15
Thom isomorphism, 23
Torsion, 97
Transport equation, 127
Transversality, 10
Tubular neighborhood, 10

U

$\mathcal{U}_\rho$, 122

V
Valuated symplectic form, 191
Valuation map, 189
Vector bundle, 6
Vector field, 8

W
WZ-system, 239